Quantitative estimates for Lévy driven SDEs with different drifts and applications

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Outline

1. Motivations
2. Main results
3. Proof

Motivating example (Sun-Xie)

Consider one-dimensional O-U process:

$$\mathrm{d}X_t = -X_t\,\mathrm{d}t + \mathrm{d}B_t.$$

It is known that X_t exponentially converges to $\pi := N(0, 1/2)$ as $t \to \infty$.

Question:

$$\mathrm{d}X_t = ((1+t)^{-1} - 1)X_t\,\mathrm{d}t + \mathrm{d}B_t.$$

It can be proved that the process X_t is W_2-strongly ergodic in the sense that

$$\lim_{t\to\infty} W_2(P(s,x;t,\cdot),\pi) = 0.$$

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Strong ergodicity:

$$\sup_{x\in\mathbb{R}} \|P(t,x,\cdot)-\pi\|_{\mathrm{Var}} \le Ce^{-\lambda t}.$$

Nonhomogeneous Markov chain: Isofescu (1980/2007): Finite Markov chains and their applications.

$$\lim_{t\to\infty} \|P(s,x;t,\cdot)-\pi\|_{\mathrm{Var}} = 0.$$

Note: In general we cannot hope to find a single invariant measure. $(\pi_s)_{s\ge 0}$ is a system of invariant measures (Da Prato-Röckner, 08):

$$\int_{\mathbb{R}} P_{s,t}f(x)\,\pi_t(\mathrm{d}x) = \int_{\mathbb{R}} f(x)\,\pi_s(\mathrm{d}x), \quad s\le t.$$

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Example: Time-dependent stable-like process

$$\mathcal{L}_t f(x) = \text{p.v.} \int_{\mathbb{R}^d} (f(x+z) - f(x))K(t,x,z)\frac{1}{|z|^{d+\alpha}}\,dz,$$

where for every $t > 0$, $K(t,\cdot,\cdot)$ is multivariate 1-periodic.

$$\sup_{x\in\mathbb{R}^d} |P_{s,t}f(x) - \mu_s(f)| \leq c_0 e^{-c_1(t-s)}\|f\|_\infty.$$

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Convergence rate

One-dimensional time-inhomogeneous process

$$\mathrm{d}X_{s,t} = (\phi(t) - X_{s,t})\,\mathrm{d}t + \mathrm{d}B_t, \quad t \geq s \geq 0, \quad X_{s,s} = x.$$

$$X_{s,t} = e^{-(t-s)}x + \int_s^t e^{-(t-u)}\phi(u)\,\mathrm{d}u + \int_s^t e^{-(t-u)}\mathrm{d}B_u.$$

$$(\delta_x P_{s,t})(\mathrm{d}y) = \frac{1}{\sqrt{\pi(1-e^{-2(t-s)})}}\exp\left(-\frac{\left(y - e^{-(t-s)}x - \int_s^t e^{-(t-u)}\phi(u)\,\mathrm{d}u\right)^2}{1-e^{-2(t-s)}}\right)\mathrm{d}y$$

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When $\lim_{t\to\infty} \phi(t) = 0$, the limit process is expected to be

$$\mathrm{d}Y_t = -Y_t\,\mathrm{d}t + \mathrm{d}B_t, \quad t \geq s, \quad Y_s = x.$$

$$\begin{aligned}\mathbb{W}_2(\delta_x P_{s,t}, \pi)^2 = {} & e^{-2(t-s)}x^2 + 2\,e^{-(t-s)}x \int_s^t e^{-(t-u)}\phi(u)\,\mathrm{d}u \\ & + \left|\int_s^t e^{-(t-u)}\phi(u)\,\mathrm{d}u\right|^2 \\ & + \frac{1}{2}\Big(1 - \big(1 - e^{-2(t-s)}\big)^{1/2}\Big)^2.\end{aligned}$$

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Long-time behavior of time-inhomogeneous SDEs

Consider one-dimensional time-inhomogeneous process:

$$\mathrm{d}X_t = (\phi(t) - X_t)\,\mathrm{d}t + \mathrm{d}Z_t,$$

where $\phi : [0, \infty) \to [0, \infty)$ and $(Z_t)_{t\geq 0}$ is a one-dimensional Lévy process.

If $\lim_{t\to\infty} \phi(t) = 0$, then it is naturally expected that the process $(X_t)_{t\geq 0}$ above enjoys the same long time behavior as that of the time-homogeneous O-U process

$$\mathrm{d}\overline{X}_t = -\overline{X}_t\,\mathrm{d}t + \mathrm{d}Z_t.$$

It is well known that the process $(\overline{X}_t)_{t\geq 0}$ admits a unique invariant probability measure, written as π, and it converges exponentially to π.

Subsequently, a spontaneous question one might ask is that, under what conditions, the transition kernel of the time-inhomogeneous process $(X_t)_{t\geq 0}$ will converge to the invariant probability measure π.

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Ergodicity of the McKean-Vlasov SDE

Consider

$$\mathrm{d}X_t = b(X_t, \mathscr{L}_{X_t})\,\mathrm{d}t + \mathrm{d}Z_t,$$

where $\mathscr{L}_{X_t}$ means the law of X_t and $(Z_t)_{t\geq 0}$ is a d-dimensional Lévy process. Due to the intervention of the measure variable, the solution process $(X_t)_{t\geq 0}$ is a nonlinear Markov process whose transition kernel may depend not only on the current state of the process but also on the current distribution of the process.

Provided that the McKean-Vlasov SDE is weakly wellposed, the weak solution $(X_t)_{t\geq 0}$ shares the same distribution as that of the corresponding decoupled SDE

$$\mathrm{d}Y_t^\mu = b(Y_t^\mu, \mu_t)\,\mathrm{d}t + \mathrm{d}Z_t, \quad \mathscr{L}_{Y_0^\mu} = \mu,$$

where $\mu_t := \mathscr{L}_{X_t}$ with the initial distribution $\mathscr{L}_{Y_0} = \mu$. That is, we have $\mathscr{L}_{X_t} = \mathscr{L}_{Y_t^\mu}$ when $\mathscr{L}_{X_0} = \mathscr{L}_{Y_0^\mu} = \mu$.

Therefore, the exploration on the McKean-Vlasov SDE amounts to the counterpart of the corresponding decoupled SDE. Note that the drifts of the decoupled SDEs are not the same once the initial distributions involved are different.

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Setting

In this talk, we are interested in the following SDEs on $\mathbb{R}^d$:

$$\mathrm{d}X_t = b_t(X_t)\,\mathrm{d}t + \mathrm{d}Z_t,$$

and

$$\mathrm{d}Y_t = \tilde{b}_t(Y_t)\,\mathrm{d}t + \mathrm{d}Z_t,$$

where $b, \tilde{b} : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ are measurable, and $(Z_t)_{t\ge 0}$ is a d-dimensional pure jump Lévy process with the Lévy measure ν.

• Eberle, A. and Zimmer, R.: Sticky couplings of multidimensional diffusions with different drifts, *Ann. Inst. Henri Poincaré Probab. Stat.*, **55** (2019), 2370–2394.

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Assumptions

For the drift $b_t(x)$ and the Lévy measure $\nu(\mathrm{d}z)$, we assume that

$(\mathbf{A}_1)$ (i) *there exist constants* $K_1, \ell_0 \geq 0$ *and* $K_2 > 0$ *such that for all* $t \geq 0$ *and* $x, y \in \mathbb{R}^d$,

$$\langle x-y, b_t(x)-b_t(y)\rangle \leq \big(K_1 1_{\{|x-y|\leq \ell_0\}} - K_2 1_{\{|x-y|>\ell_0\}}\big)|x-y|^2.$$

(ii) *there exist a constant* $\kappa > 0$ *and a nondecreasing and concave function* $\sigma \in C([0,2\ell_0];\mathbb{R}_+) \cap C^2((0,2\ell_0];\mathbb{R}_+)$ *such that* $[0,2\ell_0] \ni r \mapsto \int_0^r \frac{1}{\sigma(s)}\,\mathrm{d}s$ *is integrable, and*

$$\sigma(r) \leq \frac{1}{2r} J^\nu(r\wedge\kappa)(\kappa\wedge r)^2, \quad r \in (0,2\ell_0],$$

where

$$J^\nu(s) := \inf_{x\in\mathbb{R}^d, |x|\leq s} \big(\nu \wedge (\delta_x * \nu)\big)(\mathbb{R}^d), \quad s > 0.$$

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$(\mathbf{A}_2)$ *there exist a C^2-function $W:\mathbb{R}^d\to[0,\infty)$, locally integrable functions $\phi_1,\phi_2,\phi_3:[0,\infty)\to[0,\infty)$ and a locally integrable function $\lambda_W:[0,\infty)\to\mathbb{R}$ such that for all $t\ge 0$ and $x\in\mathbb{R}^d$,*

$$|b_t(x)-\tilde b_t(x)|\le \phi_1(t)+\phi_2(t)W(x),$$

and

$$(\mathscr{L}_t^{\tilde b}W)(x)\le \phi_3(t)+\lambda_W(t)W(x),$$

where $\mathscr{L}_t^{\tilde b}$ means the infinitesimal generator of $(Y_t)_{t\ge0}$.

Main result

Theorem 1

Assume that $(\mathbf{A_1})$ *and* $(\mathbf{A_2})$ *hold. Then, for all* $t > s$ *and* $x, y \in \mathbb{R}^d$,

$$\begin{aligned}
&\mathbb{W}_1(\delta_x P^X_{s,t}, \delta_y P^Y_{s,t}) \\
&\le \frac{1+c^*}{2c^*}\Bigg[\mathrm{e}^{-\lambda(t-s)}\,|x-y| + \int_s^t \mathrm{e}^{-\lambda(t-r)}\,\phi_1(r)\,\mathrm{d}r \\
&\qquad + \int_s^t \phi_2(r)\,\mathrm{e}^{-\lambda(t-r)}\left(\mathrm{e}^{\int_s^r \lambda_W(u)\,\mathrm{d}u}\,W(y) + \int_s^r \phi_3(u)\,\mathrm{e}^{\int_u^r \lambda_W(v)\,\mathrm{d}v}\,\mathrm{d}u\right)\mathrm{d}r\Bigg],
\end{aligned}$$

where

$$c^* := \mathrm{e}^{-g(2\ell_0)}, \quad \lambda := \frac{(1\wedge(2K_2))c^*}{1+c^*}, \quad g(2\ell_0) := (1+2K_1)\int_0^{2\ell_0}\frac{1}{\sigma(s)}\,\mathrm{d}s.$$

Corollary

Corollary 1

Consider the time-homogeneous versions of two SDEs above. Assume that $(\mathbf{A}_1)$ *and* $(\mathbf{A}_2)$ *hold with* $\lambda_W(t) \equiv \lambda_W < 0$, $\phi_1(t) \equiv \kappa_1, \phi_2(t) \equiv \kappa_2$, *and* $\phi_3(t) \equiv C_W$ *for some constants* $C_W, \kappa_1, \kappa_2 > 0$. *Then for all* $t > 0$ *and* $x, y \in \mathbb{R}^d$,

$$\mathbb{W}_1\big(\delta_x P_t^X, \delta_y P_t^Y\big) \le \frac{1+c^*}{2c^*}\Bigg\{ \mathrm{e}^{-\lambda t}\,|x-y| + \lambda^{-1}(\kappa_1 - \kappa_2 C_W/\lambda_W)(1-\mathrm{e}^{-\lambda t}) + \kappa_2\,\mathrm{e}^{\lambda_W t}\,W(y)\left[\frac{1}{\lambda+\lambda_W}(1-\mathrm{e}^{-(\lambda+\lambda_W)t})\right]\Bigg\}.$$

Application: Long time behavior of inhomogeneous SDEs with jumps

Let $(X_t)_{t\geq 0}$ be the unique strong solution to the time-inhomogeneous SDE, which fulfills Assumption $(\mathbf{A}_1)$. Assume that the following time-homogeneous SDE on $\mathbb{R}^d$:

$$\mathrm{d}\overline{X}_t = \overline{b}(\overline{X}_t)\,\mathrm{d}t + \mathrm{d}Z_t$$

with $\overline{b}:\mathbb{R}^d \to \mathbb{R}^d$ has a unique strong solution, which is denoted by $(\overline{X}_t)_{t\geq 0}$.

We assume that

$(\mathbf{C})$ *there are a C^2-function $W:\mathbb{R}^d \to [0,\infty)$ and a bounded function $\phi:[0,\infty)\mapsto[0,\infty)$ that satisfies $\lim_{t\to\infty}\phi(t)=0$, so that for all $x\in\mathbb{R}^d$ and $t\geq 0$,*

$$|b_t(x)-\overline{b}(x)| \leq \phi(t)W(x)$$

and there are constants $c_0,\theta>0$ such that

$$(\overline{\mathscr{L}}^{\overline{b}}W)(x) \leq c_0 - \theta W(x), \quad x\in\mathbb{R}^d.$$

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Application: Long time behavior of inhomogeneous SDEs with jumps

Theorem 2

Assume that $(\mathbf{A}_1)$ *and* $(\mathbf{C})$ *hold. Then, for all* $x \in \mathbb{R}^d$ *and* $t > s \geq 0$,

$$\mathbb{W}_1(\delta_x P^X_{s,t}, \pi) \leq C(x)\,\mathrm{e}^{-\lambda(t-s)} + \frac{1+c^*}{2c^*}\left[\frac{\|\phi\|_\infty W(x)\,\mathrm{e}^{-\lambda(t-s)}(\mathrm{e}^{(\lambda-\theta)(t-s)}-1)}{\lambda-\theta} + \frac{c_0}{\theta}\int_s^t \phi(r)\,\mathrm{e}^{-\lambda(t-r)}\,\mathrm{d}r\right],$$

where π *is the unique invariant probability measure of the process* $(\overline{X}_t)_{t\geq 0}$. *In particular,*

$$\mathbb{W}_1(\delta_x P^X_{s,t}, \pi) \to 0, \quad t \to \infty.$$

Furthermore, if $\phi(t) = c_1\,\mathrm{e}^{-\lambda_0 t}$ *for some constants* $c_1, \lambda_0 > 0$, *then for any* $0 < \lambda_* < \min\{\lambda, \lambda_0, \theta\}$, $x \in \mathbb{R}^d$ *and* $t > s$,

$$\mathbb{W}_1(\delta_x P^X_{s,t}, \pi) \leq C_*(x)\,\mathrm{e}^{-\lambda_*(t-s)}.$$

Application: Long time behavior of inhomogeneous SDEs with jumps

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Application: Exponential ergodicity of McKean-Vlasov SDEs with partially dissipative drift

Consider the following McKean-Vlasov SDE

$$\mathrm{d}X_t = b(X_t, \mathscr{L}_{X_t})\,\mathrm{d}t + \mathrm{d}Z_t$$

so that

$(\mathbf{B}_1)$ *the Lévy measure* ν *satisfies Assumption* $(\mathbf{A}_1)(\mathrm{ii})$ *and* $\int_{\{|z|>1\}} |z|\,\nu(\mathrm{d}z) < \infty$.

$(\mathbf{B}_2)$ *there exist constants* $K_1, \ell_0 \geq 0$ *and* $K_2 > 0$ *such that for all* $\mu \in \mathscr{P}(\mathbb{R}^d)$ *and* $x, y \in \mathbb{R}^d$,

$$\langle x-y, b(x,\mu) - b(y,\mu)\rangle \leq \left(K_1 1_{\{|x-y|\leq \ell_0\}} - K_2 1_{\{|x-y|>\ell_0\}}\right)|x-y|^2,$$

and there exists a constant $L_0 \geq 0$ *such that for all* $\mu, \nu \in \mathscr{P}_1(\mathbb{R}^d)$ *and* $x \in \mathbb{R}^d$,

$$|b(x,\mu) - b(x,\nu)| \leq L_0 \mathbb{W}_1(\mu,\nu).$$

Application: Exponential ergodicity of McKean-Vlasov SDEs with partially dissipative drift

Theorem 3

Assume that $(\mathbf{B}_1)$ *and* $(\mathbf{B}_2)$ *hold. Then, for all* $t \geq 0$ *and* $\mu, \hat{\mu} \in \mathscr{P}_1(\mathbb{R}^d)$,

$$\mathbb{W}_1(\mu_t, \hat{\mu}_t) \leq \frac{1+c^*}{2c^*} \, \mathrm{e}^{-\left(\lambda - \frac{(1+c^*)L_0}{2c^*}\right)t} \, \mathbb{W}_1(\mu, \hat{\mu}).$$

- Consider the following SDE

$$\begin{cases} \mathrm{d}\tilde{X}_t = b_t(\tilde{X}_t)\,\mathrm{d}t + \mathrm{d}Z_t \\ \mathrm{d}\tilde{Y}_t^{\tilde{\nu},\kappa} = \tilde{b}_t(\tilde{Y}_t^{\tilde{\nu},\kappa})\,\mathrm{d}t + \mathrm{d}Z_t + \int_{\mathbb{R}^d\times[0,1]} U^{\tilde{\nu}}((\tilde{X}_{t-} - \tilde{Y}_{t-}^{\tilde{\nu},\kappa}), z, u) N(\mathrm{d}t, \mathrm{d}z, \mathrm{d}u) \end{cases}$$

has a unique strong solution $(\tilde{X}_t, \tilde{Y}_t^{\tilde{\nu},\kappa})_{t\ge 0}$, where for $x, z \in \mathbb{R}^d$ and $u \in [0,1]$,

$$U^{\tilde{\nu}}(x,z,u) := x\big(1_{\{u\le \frac{1}{2}\rho^{\tilde{\nu}}(-x,z)\}} - 1_{\{\frac{1}{2}\rho^{\tilde{\nu}}(-x,z) < u \le \frac{1}{2}(\rho^{\tilde{\nu}}(-x,z)+\rho^{\tilde{\nu}}(x,z))\}}\big)$$

with $\rho^{\tilde{\nu}}(x,z) := \frac{\tilde{\nu}_x(\mathrm{d}z)}{\nu(\mathrm{d}z)} \in [0,1]$.

- Luo, D. and Wang, J.: Refined couplings and Wasserstein-type distances for SDEs with Lévy noises, *Stoch. Process. Appl.*, **129** (2019), 3129–3173.

- Liang, M., Majka, M.B. and Wang, J.: Exponential ergodicity for SDEs and McKean-Vlasov processes with Lévy noise, *Ann. Inst. Henri Poincaré Probab. Stat.*, **57** (2021), 1665–1701.

Coupling approach

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Coupling approach

- For any $t \geq s$,

$$\begin{aligned}\mathrm{d}\psi(|r_t|) = {} & \frac{\psi'(|r_t|)}{|r_t|}\langle r_t, b_t(\tilde{X}_t) - \tilde{b}_t(\tilde{Y}_t^{\nu,\kappa})\rangle 1_{\{r_t \neq \mathbf{0}\}}\,\mathrm{d}t \\ & + \frac{1}{2}\nu_{(r_t)_\kappa}(\mathbb{R}^d)\big(\psi(|r_t| + \kappa \wedge |r_t|) + \psi(|r_t| - \kappa \wedge |r_t|) - 2\psi(|r_t|)\big)\,\mathrm{d}t \\ & + \mathrm{d}\overline{\mathcal{M}}_{s,t}\end{aligned}$$

for some martingale $(\overline{\mathcal{M}}_{s,t})_{t \geq s}$.

-

$$\langle r_t, b_t(\tilde{X}_t) - \tilde{b}_t(\tilde{Y}_t^{\nu,\kappa})\rangle$$

- Coupling time does not make sense in this setting.

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Proof: application 2

Consider the McKean-Vlasov SDE

$$\mathrm{d}X_t = b(X_t, \mathscr{L}_{X_t})\,\mathrm{d}t + \mathrm{d}Z_t.$$

Denote $\mu_t := \mathscr{L}_{X_t}$ with $\mathscr{L}_{X_0} = \mu \in \mathscr{P}_1(\mathbb{R}^d)$ and $\hat{\mu}_t := \mathscr{L}_{X_t}$ with $\mathscr{L}_{X_0} = \hat{\mu} \in \mathscr{P}_1(\mathbb{R}^d)$. Consider the following two SDEs: for $\mu, \hat{\mu} \in \mathscr{P}_1(\mathbb{R}^d)$,

$$\mathrm{d}Y_t^{\mu} = b(Y_t^{\mu}, \mu_t)\,\mathrm{d}t + \mathrm{d}Z_t, \quad \mathscr{L}_{Y_0^{\mu}} = \mu,$$

and

$$\mathrm{d}Y_t^{\hat{\mu}} = b(Y_t^{\hat{\mu}}, \hat{\mu}_t)\,\mathrm{d}t + \mathrm{d}Z_t, \quad \mathscr{L}_{Y_0^{\hat{\mu}}} = \hat{\mu}.$$

Let

$$b_t(x) = b_t^{\mu}(x) := b(x, \mu_t), \qquad \tilde{b}_t(x) = b_t^{\hat{\mu}}(x) = b(x, \hat{\mu}_t), \quad x \in \mathbb{R}^d, \quad t \geq 0.$$

Thus, two SDEs above can be reformulated into our framework. The SDE above is called the decoupled SDE associated with the McKean-Vlasov SDE.

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Remark: total variation

- For any $t \geq s$,

$$\mathrm{d}\psi(|r_t|) = \frac{\psi'(|r_t|)}{|r_t|}\langle r_t, b_t(\tilde{X}_t) - \tilde{b}_t(\tilde{Y}_t^{\nu,\kappa})\rangle 1_{\{r_t \neq \mathbf{0}\}}\,\mathrm{d}t$$
$$+ \frac{1}{2}\nu_{(r_t)_\kappa}(\mathbb{R}^d)\big(\psi(|r_t| + \kappa \wedge |r_t|) + \psi(|r_t| - \kappa \wedge |r_t|) - 2\psi(|r_t|)\big)\,\mathrm{d}t + \cdots$$

- Lévy kernel $\mu(r, \mathrm{d}z)$ associated with the coupling process is given by

$$\mu(r, \mathrm{d}z) = \frac{1}{2}\mu_r(\mathbb{R}^d)\big(\delta_r(\mathrm{d}z) + \delta_{-r}(\mathrm{d}z)\big), \qquad r \geq 0.$$

- Let $\overline{\mathscr{L}}$ be the Lévy-type operator expressed as follows: for $h \in C_b^2(\mathbb{R})$ and $r \geq 0$,

$$(\overline{\mathscr{L}}h)(r) = h'(r)(\phi(r) + M) + \int_{\mathbb{R}} \big(h(r+z) - h(r)\big)\overline{\mu}(r, \mathrm{d}z).$$

where $\phi(r) := K_1 1_{\{0 \leq r \leq \ell_0\}} - K_2 r 1_{\{r > \ell_0\}}$, and the Lévy measure $\overline{\mu}$ fulfils

$$\overline{\mu}(r, [x, \infty)) \geq 1_{\{0 \leq x \leq r\}}\mu_x(\mathbb{R}^d),\ r \geq 0,\ x \geq 0; \quad \overline{\mu}(r, (-\infty, x)) = 0,\ r \geq 0,\ x < 0.$$

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Thank you!