McKean-Vlasov stochastic differential equation

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Background – Classical SDE

Consider the following stochastic differential equation: for $t > s \geqslant 0$,

$$\mathrm{d}X_{s,t} = b(t, X_{s,t})\mathrm{d}t + \mathrm{d}W_t, \quad X_{s,s} = x \in \mathbb{R}^d. \tag{0.1}$$

The solution $X_{s,t}(x)$ is a Markov process with generator

$$\mathscr{L}_t\varphi(x) := \frac{1}{2}\Delta\varphi(x) + b(t,x)\cdot\nabla\varphi(x), \quad \forall\varphi \in C_0^\infty(\mathbb{R}^d).$$

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Questions of interest:

1. Short time property – smoothing effect
2. Long time behavior – invariant measure

Consider the following partial differential equation:

$$\begin{cases} \partial_t u(t,x) + \mathscr{L}_t u(t,x) + f(t,x) = 0, \quad t \in [0,T), \\ u(T,x) = 0. \end{cases} \tag{0.2}$$

The unique solution is given by

$$\begin{aligned} u(t,x) &= \int_t^T \mathbb{E} f(s, X_{s,t}(x)) \mathrm{d}s = \int_t^T \mathcal{T}_{s,t} f(s,x) \mathrm{d}s \\ &= \int_t^T \int_{\mathbb{R}^d} p(s,x;t,y) f(s,y) \mathrm{d}y \mathrm{d}s. \end{aligned}$$

1. Short time

- The two-sided estimates:

$$p(s,x;t,y) \asymp \frac{1}{(t-s)^{d/2}} \exp\left\{ -c_0 \frac{|x-y|^2}{t-s} \right\}.$$

- The optimal regularity of the solution:

$$f \in C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d) \Longrightarrow u \in C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d).$$

2. Long time

Time homogeneous + the Lyapunov condition:

$$\mathscr{L}V(x) \leqslant C_0 - C_1 V(x), \quad C_0, C_1 > 0,$$

there exists a unique invariant measure $\mu(\mathrm{d}y)$ for $X_t(x)$.

The invariant measure is just the limit of the distribution of $X_t(x)$:

$$\lim_{t\to\infty} \mathcal{L}_{X_t(x)}(\mathrm{d}y) = \lim_{t\to\infty} p(t,x,y)\mathrm{d}y = \mu(\mathrm{d}y).$$

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Example: $$\mathrm{d}X_t = -X_t^3 + \mathrm{d}W_t, \quad X_0 = x \in \mathbb{R}^d,$$

with $$\mu(\mathrm{d}y) = \lim_{t\to\infty} p(t,x,y)\mathrm{d}y = c_d \mathrm{e}^{-|y|^4/4}\mathrm{d}y.$$

Consider the following McKean-Vlasov stochastic differential equation:

$$\mathrm{d}X_{s,t} = b(t, X_{s,t}, \mathcal{L}_{X_{s,t}})\mathrm{d}t + \mathrm{d}W_t, \quad X_{s,s} = \xi. \tag{0.3}$$

Example:

$$\mathrm{d}X_{s,t} = -X_{s,t}^3 + \lambda\mathbb{E}(X_{s,t})\mathrm{d}t + \mathrm{d}W_t, \quad X_{s,s} = \xi. \tag{0.4}$$

McKean-Vlasov equation

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The (formal) generator of (0.3) is given by

$$\begin{aligned}\mathscr{L}_t\varphi(x,\mu) := &\frac{1}{2}\Delta_x\varphi(x,\mu) + b(t,x,\mu)\cdot\nabla_x\varphi(x,\mu)\\ &+ \int_{\mathbb{R}^d}\Big[\tfrac{1}{2}\partial_{\tilde{x}}\big[\partial_\mu\varphi(x,\mu)(\tilde{x})\big] + b(t,\tilde{x},\mu)\cdot\partial_\mu\varphi(x,\mu)(\tilde{x})\Big]\mu(\mathrm{d}\tilde{x}),\end{aligned}$$

where $\partial_\mu\varphi$ is the Lion's derivative.

Question 1: short time

Consider the Cauchy problem on $[0, T] \times \mathbb{R}^d \times \mathscr{P}(\mathbb{R}^d)$:

$$\begin{cases} \partial_t U(t,x,\mu) + \mathscr{L}_t U(t,x,\mu) + f(t,x,\mu) = 0, \quad t \in [0,T), \\ U(T,x,\mu) = 0. \end{cases} \tag{0.5}$$

The unique solution is given by

$$\begin{aligned} u(t,x,\mu) &= \int_t^T \mathbb{E} f(s, X_{s,t}(x,\xi), \mathcal{L}_{X_{s,t}(\xi)}) \mathrm{d}s \\ &= \int_t^T \int_{\mathbb{R}^d} p(\mu; s, x; t, y) f(s, y, \mathcal{L}_{X_{s,t}(\xi)}) \mathrm{d}y \mathrm{d}s, \qquad \xi \sim \mu. \end{aligned}$$

- **Aim:** • Sharp two-sided estimates of the density function.
  • Optimal regularity of the solution w.r.t. the measure argument.

Question 2: long time

Consider the following equation:

$$\mathrm{d}X_t = -X_t^3 \mathrm{d}t + \lambda \mathbb{E} X_t \mathrm{d}t + \mathrm{d}W_t, \quad X_0 = x \in \mathbb{R}^d. \tag{0.6}$$

We know that:

(1) there exists exactly three invariant measure μ_1, μ_2 and μ_3;

(2) for any initial point x, as $t \to \infty$, the distribution $\mathcal{L}_{X_t}$ converge to one of μ_1, μ_2 and μ_3.

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▶ **Question:**

which one does it converge to? (the domain of attraction of the invariant measures)

Question 2: long time

Since for some $i = 1, 2, 3$,

$$\lim_{t\to\infty} \mathcal{L}_{X_t}(\mathrm{d}y) = \mu_i \quad \text{and} \quad \mathcal{L}_{X_t}(\mathrm{d}y) = p(t, x, y)\mathrm{d}y,$$

the question can be transferred to study

$$\lim_{t\to\infty} p(t, x, y) = ?$$

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- the drift b: Kato class + linear growth $\Longrightarrow$ density estimate/short time
- the drift b: Lyapunov condition $\Longrightarrow$ invariant measure/long time

Question 2: long time

Classical SDE with super-linear growth drift: for $n \geqslant 1$,

$$\mathrm{d}X_t = -X_t^{2n+1} + \mathrm{d}W_t, \quad X_0 = x \in \mathbb{R}^d.$$

- Long time limit ✓
- Short time density estimate ?

Thank You !