**Kean-Vlasov stochastic differential equation

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 QQ

Consider the following stochastic differential equation: for $t > s \geq 0$,

$$
dX_{s,t} = b(t, X_{s,t})dt + dW_t, \quad X_{s,s} = x \in \mathbb{R}^d.
$$
 (0.1)

The solution $X_{s,t}(x)$ is a Markov process with generator

[D](#page-0-0)raft ^Ltϕ(x) := ¹ 2 ∆ϕ(x) + b(t, x) · ∇ϕ(x), ∀ϕ ∈ C ∞ 0 (R d).

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Questions of interest:

- 1. Short time property smoothing effect
- 2. Long time behavior invariant measure

Consider the following partial differential equation:

$$
\begin{cases}\n\partial_t u(t,x) + \mathscr{L}_t u(t,x) + f(t,x) = 0, & t \in [0, \mathcal{T}), \\
u(\mathcal{T}, x) = 0.\n\end{cases}
$$
\n(0.2)

The unique solution is given by

the following partial differential equation:

\n
$$
\begin{cases}\n\frac{\partial_t u(t, x) + \mathcal{L}_t u(t, x) + f(t, x) = 0, & t \in [0, T), \\
u(T, x) = 0.\n\end{cases}
$$
\nthe solution is given by

\n
$$
u(t, x) = \int_t^T \mathbb{E}f(s, X_{s,t}(x))ds = \int_t^T \mathcal{T}_{s,t}f(s, x)ds
$$
\n
$$
= \int_t^T \int_{\mathbb{R}^d} p(s, x; t, y)f(s, y)dyds.
$$
\nwhere x is (JSNU)

\nwhere x is the function of the function y is the function of the function y is the function <

• The two-sided estimates:

[D](#page-0-0)raft p(s, x;t, y) 1 (t − s) d/2 exp n − c⁰ |x − y| 2 t − s o .

• The optimal regularity of the solution:

$$
f\in C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)\Longrightarrow u\in C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d).
$$

2. Long time

Time homogeneous $+$ the Lyapunov condition:

$$
\mathscr{L}V(x)\leqslant C_0-C_1V(x),\quad C_0,C_1>0,
$$

there exists a unique invariant measure $\mu(dy)$ for $X_t(x)$.

The invariant measure is just the limit of the distribution of $X_t(x)$:

$$
\begin{aligned}\n\text{g time} \\
\text{mogeneous } + \text{ the Lyapunov condition:} \\
\mathscr{L}V(x) &\leq C_0 - C_1 V(x), \quad C_0, C_1 > 0, \\
\text{sts a unique invariant measure } \mu(\text{d}y) \text{ for } X_t(x). \\
\text{inm } \mathcal{L}_{X_t(x)}(\text{d}y) &= \lim_{t \to \infty} p(t, x, y) \text{d}y = \mu(\text{d}y). \\
\text{where } \mathcal{L}_{X_t(x)}(x) &= \lim_{t \to \infty} p(t, x, y) \text{d}y = \mu(\text{d}y).\n\end{aligned}
$$
\n
$$
\text{McKean-Vlassov stochastic equation}
$$
\n
$$
\begin{aligned}\n\text{Fek } 26, 20 \\
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$$

2. Long time

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The invariant measure is just the limit of the distribution of $X_t(x)$:

$$
\lim_{t\to\infty}\mathcal{L}_{X_t(x)}(\mathrm{d}y)=\lim_{t\to\infty}p(t,x,y)\mathrm{d}y=\mu(\mathrm{d}y).
$$

Example:

$$
dX_t = -X_t^3 + dW_t, \quad X_0 = x \in \mathbb{R}^d,
$$

Figure

\nmggeneous + the Lyapunov condition:

\n
$$
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\nsts a unique invariant measure $\mu(\mathrm{d}y)$ for $X_t(x)$.

\niriant measure is just the limit of the distribution of $X_t(x)$:

\n
$$
\lim_{t \to \infty} \mathcal{L}_{X_t(x)}(\mathrm{d}y) = \lim_{t \to \infty} p(t, x, y) \mathrm{d}y = \mu(\mathrm{d}y).
$$
\nif $\mathrm{d}X_t = -X_t^3 + \mathrm{d}W_t, \quad X_0 = x \in \mathbb{R}^d$,

\nwith $\mu(\mathrm{d}y) = \lim_{t \to \infty} p(t, x, y) \mathrm{d}y = c_d e^{-|y|^4/4} \mathrm{d}y.$

\nif $\mathrm{d}X_t(x) = \frac{1}{\lambda} \int_{-\lambda}^{\lambda} \mathcal{L}_{\lambda}(x) \mathrm{d}x$.

\nwhere λ is a constant, λ is a constant.

Consider the following McKean-Vlasov stochastic differential equation:

$$
dX_{s,t} = b(t, X_{s,t}, \mathcal{L}_{X_{s,t}})dt + dW_t, \quad X_{s,s} = \xi.
$$
 (0.3)

Example:

n-Vlasov equation
\nthe following McKean-Vlasov stochastic differential equation:
\n
$$
dX_{s,t} = b(t, X_{s,t}, \mathcal{L}_{X_{s,t}})dt + dW_t, \quad X_{s,s} = \xi.
$$
\n(0.3)
\n
$$
dX_{s,t} = -X_{s,t}^3 + \lambda \mathbb{E}(X_{s,t})dt + dW_t, \quad X_{s,s} = \xi.
$$
\n(0.4)
\nWeKear-Vlasov stochastic equation
\n
$$
dX_{s,t} = \sum_{s=0}^{s} \lambda_s \mathbb{E}(X_{s,t})dt + dW_t, \quad X_{s,s} = \xi.
$$

Consider the following McKean-Vlasov stochastic differential equation:

$$
dX_{s,t} = b(t, X_{s,t}, \mathcal{L}_{X_{s,t}})dt + dW_t, \quad X_{s,s} = \xi.
$$
 (0.3)

Example:

$$
dX_{s,t} = -X_{s,t}^3 + \lambda \mathbb{E}(X_{s,t})dt + dW_t, \quad X_{s,s} = \xi.
$$
 (0.4)

The (formal) generator of (0.3) is given by

McKean-Vlasov equation	
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$dX_{s,t} = b(t, X_{s,t}, \mathcal{L}_{X_{s,t}})dt + dW_t, \quad X_{s,s} = \xi.$	(0.3)
Example:	
$dX_{s,t} = -X_{s,t}^3 + \lambda \mathbb{E}(X_{s,t})dt + dW_t, \quad X_{s,s} = \xi.$	(0.4)
The (formal) generator of (0.3) is given by	
$\mathcal{L}_t\varphi(x,\mu) := \frac{1}{2}\Delta_x\varphi(x,\mu) + b(t,x,\mu) \cdot \nabla_x\varphi(x,\mu)$	
$+ \int_{\mathbb{R}^d} \left[\frac{1}{2} \partial_{\bar{x}} \left[\partial_\mu \varphi(x,\mu)(\bar{x}) \right] + b(t, \bar{x}, \mu) \cdot \partial_\mu \varphi(x,\mu)(\bar{x}) \right] \mu(d\bar{x}),$	
where $\partial_\mu \varphi$ is the Lion's derivative.	
Longie Xie (JSNU)	McKean-Vlasov stochastic equation

where $\partial_{\mu}\varphi$ is the Lion's derivative.

Consider the Cauchy problem on $[0,T]\times \mathbb{R}^d\times \mathscr{P}(\mathbb{R}^d)$:

$$
\begin{cases}\n\partial_t U(t, x, \mu) + \mathscr{L}_t U(t, x, \mu) + f(t, x, \mu) = 0, \quad t \in [0, T), \\
U(T, x, \mu) = 0.\n\end{cases}
$$
\n(0.5)

The unique solution is given by

station 1: short time

\nsideer the Cauchy problem on
$$
[0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)
$$
:

\n
$$
\begin{cases}\n\partial_t U(t, x, \mu) + \mathcal{L}_t U(t, x, \mu) + f(t, x, \mu) = 0, & t \in [0, T), \\
U(T, x, \mu) = 0.\n\end{cases}
$$
\nunique solution is given by

\n
$$
u(t, x, \mu) = \int_t^T \mathbb{E}f(s, X_{s,t}(x, \xi), \mathcal{L}_{X_{s,t}(\xi)}) \, \mathrm{d}s
$$
\n
$$
= \int_t^T \int_{\mathbb{R}^d} p(\mu; s, x; t, y) f(s, y, \mathcal{L}_{X_{s,t}(\xi)}) \, \mathrm{d}y \, \mathrm{d}s, \qquad \xi \sim \mu.
$$
\nim: **6** Sharp two-sided estimates of the density function.

\nOptimal regularity of the solution w.r.t. the measure argument

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\nMcKean-Ylasov stochastic equation

\nLet 2π , 2π , 2π , and π

- $Aim: \bullet$ Sharp two-sided estimates of the density function.
	- Optimal regularity of the solution w.r.t. the measure argument.

Consider the following equation:

$$
dX_t = -X_t^3 dt + \lambda \mathbb{E} X_t dt + dW_t, \quad X_0 = x \in \mathbb{R}^d.
$$
 (0.6)

We know that:

- (1) there exists exactly three invariant measure μ_1 , μ_2 and μ_3 ;
- on 2: long time

the following equation:
 $dX_t = -X_t^3 dt + \lambda \mathbb{E}X_t dt + dW_t, \quad X_0 = x \in \mathbb{R}^d.$

or that:

ere exists exactly three invariant measure μ_1 , μ_2 and μ_3 ;

r any initial point x, as $t \to \infty$, the distribut (2) for any initial point x , as $t\rightarrow\infty$, the distribution \mathcal{L}_{X_t} converge to one of μ_1 , μ_2 and μ_3 .

Consider the following equation:

$$
dX_t = -X_t^3 dt + \lambda \mathbb{E} X_t dt + dW_t, \quad X_0 = x \in \mathbb{R}^d.
$$
 (0.6)

We know that:

- (1) there exists exactly three invariant measure μ_1 , μ_2 and μ_3 ;
- (2) for any initial point x , as $t\rightarrow\infty$, the distribution \mathcal{L}_{X_t} converge to one of μ_1 , μ_2 and μ_3 .

▶ Question:

on 2: long time

the following equation:
 $dX_t = -X_t^3 dt + \lambda \mathbb{E}X_t dt + dW_t, \quad X_0 = x \in \mathbb{R}^d.$

or that:

ere exists exactly three invariant measure μ_1 , μ_2 and μ_3 ;

r any initial point x, as $t \to \infty$, the distribut which one does it converge to? (the domain of attraction of the invariant measures)

Since for some $i = 1, 2, 3$,

on 2: long time
\n
$$
\begin{aligned}\n\text{some } i &= 1, 2, 3, \\
\lim_{t \to \infty} \mathcal{L}_{X_t}(\text{d}y) &= \mu_i \quad \text{and} \quad \mathcal{L}_{X_t}(\text{d}y) = p(t, x, y) \text{d}y, \\
\lim_{t \to \infty} p(t, x, y) &= ?\n\end{aligned}
$$
\n
$$
\begin{aligned}\n\lim_{t \to \infty} p(t, x, y) &= ?\n\end{aligned}
$$
\n
$$
\begin{aligned}\n\text{AEx}_{\text{R}}(J \text{SNU}) &\longrightarrow \text{AEx}_{\text{R}}(J \text{SNU})\n\end{aligned}
$$
\n
$$
\begin{aligned}\n\text{AEx}_{\text{R}}(J \text{SNU}) &\longrightarrow \text{AEx}_{\text{R}}(J \text{SNU})\n\end{aligned}
$$

the question can be transferred to study

 $\lim_{t\to\infty} p(t,x,y) = ?$

Since for some $i = 1, 2, 3$,

From
$$
i = 1, 2, 3
$$
,

\n
$$
\lim_{t \to \infty} \mathcal{L}_{X_t}(\mathrm{d}y) = \mu_i \quad \text{and} \quad \mathcal{L}_{X_t}(\mathrm{d}y) = p(t, x, y) \mathrm{d}y,
$$
\ntion can be transferred to study

\n
$$
\lim_{t \to \infty} p(t, x, y) = ?
$$
\nfit b: Kato class + linear growth \implies density estimate/short

\nfit b: Lyapunov condition \implies invariant measure/long time

\nif $\lim_{t \to \infty} \mathcal{L}_{X_t}(\mathrm{d}y) = \lim_{t \to \infty} \mathcal{L}_{X_t}(\mathrm{d}y) =$

the question can be transferred to study

$$
\lim_{t\to\infty}p(t,x,y)=?
$$

- the drift b: Kato class + linear growth \implies density estimate/short time
- the drift b: Lyapunov condition \implies invariant measure/long time

S[D](#page-0-0)E with super-linear growth drift: for $n \ge 1$,
 $dX_t = -X_t^{2n+1} + dW_t$, $X_0 = x \in \mathbb{R}^d$.

ime limit \checkmark

time density estimate ?

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McKean-Vlasov stochastic equation

Eb. 26, 202 Classical SDE with super-linear growth drift: for $n \geq 1$,

$$
dX_t = -X_t^{2n+1} + dW_t, \quad X_0 = x \in \mathbb{R}^d.
$$

- Long time limit \checkmark
- Short time density estimate?

$\begin{minipage}{0.5\textwidth} \begin{tabular}{@{}l@{}} \textbf{T} & \textbf{hank} & \textbf{You} & \textbf{I} \end{tabular} \end{minipage}$
 $\begin{minipage}{0.5\textwidth} \begin{tabular}{@{}l@{}} \textbf{A} & \textbf{B} & \textbf{B$ Thank You !

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