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HUA-CHEN NEW THEORY OF ECONOMIC OPTIMIZATION

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Abstract. Between 1957 and 1985, the Chinese mathematician Loo-Keng Hua pioneered economic optimization theory through three key contributions: establishing economic stability's fundamental theorem, proving the uniqueness of equilibrium solutions in economic systems, and developing a consumption-integrated model 50 days before his death. Since 1988, Mu-Fa Chen has been working on Hua's theory. He introduced stochastics, namely Markov chains, to economic optimization theory. He updated and developed Hua's model, and came up with a new model (Chen's model), which has become the starting point of a new economic optimization theory. Chen's theory can be applied to economic stability testing, bankruptcy prediction, product ranking and classification, economic prediction and adjustment, and economic structure optimization. Chen's theory can also provide efficient algorithms that are programmable and intelligent. Stochastics is the cornerstone of Chen's theory. There is no overlap between Chen's theory, and the existing mathematical economy theory and the economics developments that were awarded Nobel Prizes in Economics between 1969 and 2024. Chen's theory distinguishes itself from existing theories for being quantitative, calculable, predictable, optimizable, programmable, and intelligenced. This survey provides a theoretical overview of the newly published monograph [7]. Specifically, the invariant of the economic structure matrix, also known as the Chen's invariant, is first published in this survey.

1. **Introduction.** From the beginning of the last century to the 1960s, mathematics was in the Hilbert's axiomatization era, with a focus on the construction of the foundations of various branches and on the solutions of important problems. In the 1960s, since most of the existing branches of mathematics had become mature, there was a need to open new fields, and so mathematics experienced a revolution and returned to the Poincaré era: the era of integrating mathematics with various other disciplines such as physics and biology. One representative of this revolution is the random field theory, developed by the Dobrushin's school at the intersection of probability and equilibrium statistical physics. Influenced by this, starting from 1980s, Mu-Fa Chen, along with his team at Beijing Normal University, built a new research direction, specifically the infinite-dimensional reaction-diffusion processes,

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at the interaction of probability theory and non-equilibrium statistical physics, and published the research monograph [1]. In the late 1980's, Chen and his team expanded their direction. On the theoretical side, they started a systematic study of stabilitity along with the associated convergence rate and spectrum theory. On the application side, they developed and revitalized Hua's economic optimization theory. Recently, great progress has been made in the applications of mathematics, which has become key to many fields (such as weather forecasting, 3D printing, minimally invasive surgery, drones, robots, and booming artificial intelligence), triggering profound changes in many of them. All the applications mentioned above involve data collection, storage, rapid analysis, intelligent decision-making, and rapid implementation. All these applications are supported by mathematical theory.

Applying mathematics to the economy, providing mathematical support for formulating national economic development plans, and establishing a set of in-depth methods to serve national economic development are the main parts of Hua's research in economic optimization, which is a model of interdisciplinary research.

The materials presented here on Chen's invariants for economic structure matrices are completely new and have not been published elsewhere.

2. **Input-output model and preliminaries.**

2.1. **Input-output model.** In this survey, a "product's vector" (abbrev. d-vector) is regarded as an industry consisting of certain products, or a department consisting of some industries. We use the d-vector

$$x = \left(x^{(1)}, \cdots, x^{(d)}\right)$$

to denote the vector of the economic products we are concerned about, where $x^{(i)}$, $i = 1, \cdots, d$, denotes the amount of the i-th product with fixed dimension.

To understand the current economic situation, we need the following data:

- d-vector of last year: $x_0 = \left(x_0^{(1)}, \cdots, x_0^{(d)}\right)$;
- d-vector of this year: $x_1 = \left(x_1^{(1)}, \cdots, x_1^{(d)}\right)$;
- The structure matrix: $A_0 = \left(a_{ij}^{(0)} : i, j = 1, 2, \cdots, d\right)$,

where the meaning of A_0 is as follows: to produce a unit of the i-th product, $a_{ij}^{(0)}$ units of the j-th products ($j = 1, \ldots, d$) are consumed. Thus we have

$$x_0^{(j)} = \sum_{i=1}^{d} x_1^{(i)} a_{ij}^{(0)},$$

in matrix form, the above can be written as

$$x_0 = x_1 A_0.$$

Conversely, given x_0 and A_0, to uniquely determine x_1, one needs A_0 to be invertible. Assume for now that all products produced are used for reproduction. This leads to the **Idealized (no consumption) model**:

$$x_{n-1} = x_n A_{n-1},\ n \geqslant 1.$$

Hence

$$x_0 = x_1 A_0 = x_2 A_1 A_0 = \cdots = x_n A_{n-1} \cdots A_0.$$

In the time-homogeneous case $A_n \equiv A$, we get the **Input-Output model**:

$$x_0 = x_n A^n, \qquad n \geqslant 1. \tag{1}$$

If we are only concerned with stability, that is, the limit behavior on the right side of the equation above as $n \to \infty$, we only need the asymptotic behavior of A^n, and A does not need to be invertible. When A is reversible, (1) is equivalent to

$$x_n = x_0 A^{-n}, \qquad n \geqslant 1. \tag{2}$$

2.2. **Two fundamental results of matrix theory.** To state the main results of Hua's theory, we need two fundamental results in matrix theory. This section only involves two concepts: irreducibility and aperiodicity. Let $E = \{1, 2, \cdots, d\}$, $A = (a_{ij} : i, j \in E)$, $A^n = \left(a_{ij}^{(n)} : i, j \in E\right)$. In this survey, by a non negative matrix, we mean a matrix with all its entries being non negative. A positive matrix is defined similarly.

Definition 2.1. A non-negative matrix $A = (a_{ij})$ is said to be irreducible if for any pair $i, j \in E$, $i \neq j$, there exist distinct $i = i_1, i_2, \cdots, i_m = j$ such that

$$a_{i_1 i_2} > 0,\ a_{i_2 i_3} > 0,\ \cdots,\ a_{i_{m-1} i_m} > 0.$$

A being irreducible means that any pair of $i, j \in E$ is **interconnected**, that is, there is a path from i to j and there is also a path from j to i. If we put an edge between $i, j \in E$, $i \neq j$, whenever $a_{ij} > 0$, then we get a connected graph when A is irreducible. The following theorem is the most important result on non negative irreducible matrices, and it is the cornerstone of Hua's economic optimization theory. It is also a major difference between Hua's theory and the current popular input-output theory.

Theorem 2.2 (Perron-Frobenius 1907, 1912)**.** *If A is a non negative irreducible matrix, then its spectral radius $\rho(A)$ is a positive simple eigenvalue of A, and the corresponding left- and right-eigenvectors can be chosen to be positive.*

Note that $\rho(A)$ is the largest eigenvalue of A, and it is called the maximal eigenvalue of A. Let u (row vector) and v (column vector) be the corresponding left and right positive eigenvectors of A:

$$uA = \rho(A)u, \qquad Av = \rho(A)v.$$

u and v are called the maximal left and right eigenvectors of A. They are unique up to a multiplicative constant.

Based on these facts, the triplet $(\rho(A), u, v)$ is called the three major characteristics of A. As seen above, the row and column attributes of a vector can be often identified from the context , so we will not explicitly indicate them. From now on, we assume that, for any $i \in E$,

$$\left\{n \geqslant 1 : a_{ii}^{(n)} > 0\right\} \neq \emptyset. \tag{3}$$

Definition 2.3. The period of $i \in E$, d_i, is defined to be the greatest common divisor of the elements in (3). If $d_i = 1$, we say that i is **aperiodic**. When A is irreducible, there is a common period for all $i \in E$, which is called the period of A ([6] Theorem 1.26). If the period is 1, then we say that A is aperiodic.

If a non negative matrix A has a positive diagonal entry, then it is aperiodic. The following results will be useful.

Proposition 2.4. *If A is a non negative irreducible, aperiodic matrix, then exists a natural number $M \leqslant (d-1)^2 + 1$ such that A^m is a positive matrix when $m \geqslant M$* ([12] *Example* 8.3.4 *and Exercise* 8.3.9)*. If all diagonal elements are positive, the conclusion can be strengthened to $M \leqslant d - 1$* ([12] *equation* (8.3.5))*.*

Let $M_{\min}$ be the smallest M satisfying the conclusion of the proposition above.

Proposition 2.5. *If A is a non negative irreducible and aperiodic matrix, then the modulus of every eigenvalue of A different from $\rho(A)$ is less than $\rho(A)$.*

With the exception of Proposition 2.4, the material in this section is classical and well known. For details, see [6] (Chapter 1), [11] or the entry "Perron-Frobenius Theorem" in Wikipedia. The terminology used here is from the theory of Markov chains. In matrix theory, "irreducible" is called "indivisible", and a "irreducible, aperiodic, non-negative square matrix" is referred to as a "prime square matrix".

We end this section with an example showing that the invertibility assumption of A in the model is necessary. Given a positive probability distribution π (row vector) on $\{0,1\}$, and let $A = \mathbb{1}\pi$, where $\mathbb{1}$ is a column vector with all components equal to 1, then A is irreducible and aperiodic, and the only stationary distribution is π: $\pi = \pi A$. If $x_0 = \pi$, then it is easy to check that any probability distribution x_1 satisfies the equation $x_0 = x_1 A$. By assumption $A = \mathbb{1}\pi$ and the condition that $x_1\mathbb{1} = 1$, it follows that

$$x_1 A = x_1 \mathbb{1}\pi = \pi.$$

Hence, the equation $x_0 = x_1 A$ has no solution once $x_0 \neq \pi$. This indicates that this equation cannot completely determine an input-output system. The reason is that the rank of A is 1 and A is not invertible.

3. **Hua's fundamental theorem of economic optimization.** From now on, unless explicitly stated otherwise, we will assume that A is non-negative, irreducible, and aperiodic.

For a d-vector x, we write $\min_k x = \min\{x^{(1)}, \cdots, x^{(d)}\}$. For positive input d-vectors x (x has no zero-component) and y, we define the quotient $y/x = (y^{(1)}/x^{(1)}, \cdots, y^{(d)}/x^{(d)})$. For a given input d-vector x, let y be the d-vector xA, then the corresponding quotient is $y/x = xA/x$. The following lemma gives a variational formula for the maximal eigenvalue $\rho(A)$ of A using xA/x.

Lemma 3.1 (Collotz-Wielandt (C–W) formula)**.** *If A is a non negative irreducible matrix, then*

$$\max_{x>0} \min_k \frac{xA}{x} = \rho(A) = \min_{x>0} \max_k \frac{xA}{x}, \tag{4}$$

where $\max_k \frac{xA}{x} = \max_k \frac{(xA)^{(k)}}{x^{(k)}}$.

The equalities above remain valid if xA is replaced by Ax. Two equalities hold simultaneously if and only if x is the corresponding maximal eigenvector.

In applications, we often use the C-W formula (4) instead of the Rayleigh entropy method to estimate $\rho(A)$. This formula is universal, that is, for each positive vector x, there are lower and upper bound estimates:

$$\min_k \frac{xA}{x} \leqslant \rho(A) \leqslant \max_k \frac{xA}{x}.$$

If x_0 and x_1 are the input and output indices, respectively, we define $\min_k x_1/x_0$ to be the development rate of economy. The following result is Hua's fundamental theorem in economic optimization.

Theorem 3.2 (Hua's Fundamental Theorem in Economic Optimization)**.** *Let $x_{n-1} = x_n A$, $n \geqslant 1$, $\rho(A)$, u, v be the maximal left and right eigenvectors of A. Then the following conclusions hold.*

(i) *The optimal input $x_0 = u$, which gives the optimal development rate $1/\rho(A)$.*

(ii) *If A is aperiodic and $x_0 \neq u$, then the economic system will inevitably collapse, that is,*

$$\textit{collapse time}\, T_{x_0}^{+} = \inf\{n\colon \textit{a certain component of } x_n < 0\} < \infty.$$

Remark 3.3. This theorem says that u is optimal from two different perspectives.

- From the perspective of economic development rate, the optimal input is the left eigenvector u, that is, $x_0 = u$;
- In order for the economic system to never collapse, the optimal input must also be the left eigenvector u. That is, $T_{x_0}^{+} = \infty \Longrightarrow x_0 = u$.

The following is the proof given by Chen in September of 1989 for the case when A is a transition probability matrix.

Proof. (i) The slowest development rate of the economy is

$$\min_k \frac{x_0 A}{x_0}.$$

To attain the optimal development rate, we need to look for x_0 so that the quantity above achieves a maximum. It follows from Lemma 3.1 that the maximum is achieved when x_0 is the maximal left eigenvector u of A and the optimal development rate is $1/\rho(A)$.

Assume $A = P$ is a transition probability matrix. Then $P\mathbb{1} = \mathbb{1}$, that is, the maximal right eigenpair of P is $(1, \mathbb{1})$, and the optimal development rate is 1.

(ii) Assume P is irreducible and aperiodic. Then the ergodic theorem of Markov chains ([6] Theorem 1.19)

$$P^n \to \mathbb{1}\pi, \quad n \to \infty,$$

where $\pi = (\pi^{(1)}, \cdots, \pi^{(d)})$ $(\pi^{(i)} > 0, i = 1, \cdots, d)$ is the invariant probability measure of P.

Assume $\mu_0 = \mu_n P^n, \forall n \geqslant 1$, $\mu_0 \geqslant 0$ satisfies $\mu_0 \mathbb{1} = 1$. It follows from $P\mathbb{1} = \mathbb{1}$ that

$$\mu_0 \mathbb{1} = \mu_n P^n \mathbb{1} = \mu_n \mathbb{1} = 1, \ \forall n.$$

Combining this with the non-negative boundedness of $\{\mu_n\}$, we get that there exists a convergent subsequence $\{\mu_{n_k}\}_{k \geqslant 1}$. If $\mu_{n_k} \to \bar{\mu}$, then $\bar{\mu}\mathbb{1} = 1$. Therefore

$$\mu_0 = \lim_{k \to \infty} \mu_{n_k} P^{n_k} = \bar{\mu}\mathbb{1}\pi = \pi.$$

□

For a general structure matrix A, is it possible to transform A into a transition probability matrix P and use it to prove Hua's fundamental theorem. This leads to **Chen's new theory of economic optimization**.

It is well known that equilibrium is the core element of the economy. If the balance is destroyed, it may not be restored for a hundred years. In the 1990s, virtual economy developed rapidly due to the rise of the internet, and countless financial products emerged. When most of these virtual products defaulted, the financial world collapsed quickly, causing a series of collapses in the real economy. When the financial crisis broke out in 2008, Iceland declared national bankruptcy.

People often say that "balance is temporary, and imbalance is the norm". In a rigorous mathematical framework, balance or equilibrium means a certain ratio between products. The equilibrium solution is unique up to a multiplicative constant. In practice, it is important to allow this free factor. No matter how many products

Consumption / Input / Output	Agriculture	Manufacturing
Agriculture	0.25	0.14
Manufacturing	0.4	0.12

an economic model has, the equilibrium solution is just a ratio between products. The importance of the equilibrium solution lies in that it is the only stable solution of the economic system, that is, starting from it, we get $x_1 = x_0/\rho(A)$, the d-vector of each step is $1/\rho(A)$ times that of the previous step. $1/\rho(A)$ is uniquely determined by the economic system and it is the optimal development rate. In other words, the output of each step is an equilibrium solution, so it is also called a stable solution. This is the first major result of Hua's theory. Most of the results on economic optimization before Hua were existential rather than constructive, and Hua's results are computable.

Hua's most surprising result is part (ii) of Theorem 3.2. In the ideal situation without consumption, if the system does not start from the equilibrium solution, the economy will inevitably have products with different signs at a certain step (year), that is, the economy will collapse or go bankrupt. We will prove that the result remains true in the general model with consumption.

The following is the simplest model derived from Wassily Leontief, which Hua has simplified.

The ancient economy of men farming and women weaving. In this economy model, there are only two sectors (products): agriculture and manufacturing. We fix product units such as buckets and liters in agriculture, feet and yards in manufacturing, etc. The following table gives the rules of economic operation in this model.

The first row in the table says that to produce 1 unit of agricultural product, 0.25 unit of agricultural product and 0.14 unit of manufacturing product are consumed. Similarly, the second row indicates that to produce 1 unit of manufacturing product, 0.4 unit of agricultural product and 0.12 unit of manufacturing product are consumed.

The matrix composed of the four numbers in the table is called the "consumption coefficient matrix" or the "structure matrix".

$$A = \frac{1}{100}\begin{bmatrix} 25 & 14 \\ 40 & 12 \end{bmatrix} = \begin{bmatrix} 0.25 & 0.14 \\ 0.4 & 0.12 \end{bmatrix},$$

Write the output d-vector of the n-th year as (x_n, y_n), where x_n is the output of agriculture in the n-th year, and y_n is the output of manufacturing in the n-th year. Write the initial input index as (x_0, y_0). How to choose the input appropriately is the core of this theory.

The first contribution of Hua in economic optimization is finding out that the input-output table has a unique (up to a multiplicative constant) equilibrium solution or stable solution, which is the maximal left-eigenvector of A.

$$\begin{aligned} u &= (\text{agriculture},\ \ \text{manufacturing}) \\ &= \left(\frac{5}{7}\left(\sqrt{2409}+13\right),\ 20\right) \doteq (44.34397483,\ 20), \end{aligned}$$

where $\doteq$ represents approximation. Here, uniqueness is up to a multiplicative constant. In practice, any multiple of u can be used. This is of course necessary in practice: The number of products used by a small enterprise and a large enterprise can belong to different orders of magnitude. In the economic model we are studying, the input is allowed to differ by a multiplicative constant. Hua proved that if $x_0 = u$ is selected as the input, then the economic system will always remain in balance, and growth rate will be optimal. He called this method the "positive eigenvector method" (the "positive eigenvector" here is the equilibrium solution u mentioned above), which is a "proportional, high-speed development" method. Hua's most brilliant conclusion is that if $x_0 \neq u$ is selected as the input, then in a certain year, the d-vector of the economy will definitely have components with different signs. The phenomenon is commonly referred to collapse or bankruptcy. We will use T to denote the first year in which this situation occurs and call T the collapse time or bankruptcy time. We will call the few years before T as crisis times. If the collapse time is very far away, we do not need to worry too much. The surprising thing is that the system is very sensitive. Starting from the initial input (x_0, y_0), we can get the output $\{(x_n, y_n)\}_{n\geqslant 1}$ successively. We give two examples here and find out their bankruptcy and crisis times.

Example 3.4. Set the initial input value to be the equilibrium solution rounded to 3 decimal places: $(x_0, y_0) = (44.344, 20)$, and the result is shown in Figure 1 below. The blue line represents agriculture, and the red line represents manufacturing. As can be seen from the figure, the first 7 years (steps) ran well, but the increase in the 6th and 7th steps was a bit too fast, leading to a collapse in the next step, $T = 8$. For this initial value, the five-year plan is feasible.

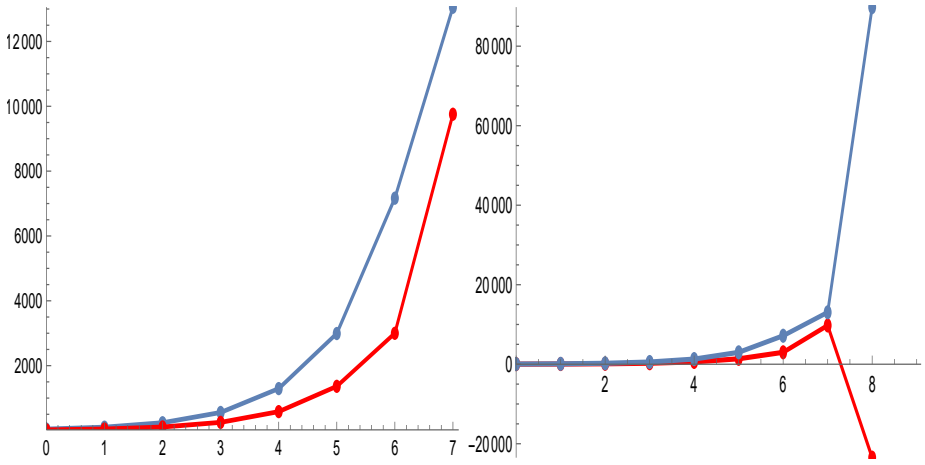


FIGURE 1.

Example 3.5. Set the initial input value to be the equilibrium solution rounded to 8 decimal places: $(x_0, y_0) = (44.34397483,\ 20)$, as shown in Figure 2, and the collapse time is $T = 13$. For this example, the precision used is already very high, which is not easy to achieve in practice, but it can only run for 12 years at most. In the 13th year, the maximum value of the upper curve reaches ten of millions, and

the minimum value of the lower curve reaches negative millions. It is easy to see that the crisis time is 10 or 11.

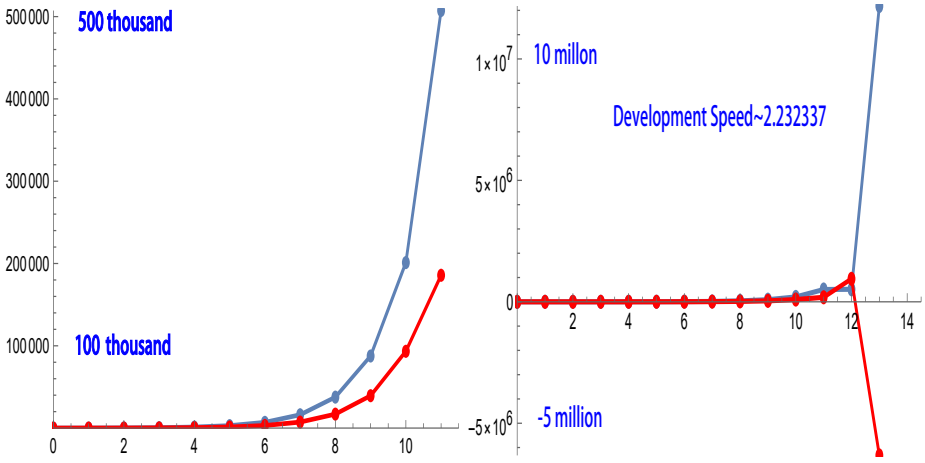


FIGURE 2.

The left-side graphs in Figure 1 and Figure 2 above stop at the crisis time, and the operation before this is relatively stable. If you go forward one or two steps, the system collapses, forming the right-side graphs. The scale of the right-side graph is highly compressed, which makes it very different from the left-side graph. The two graphs do not have the same order of magnitude. These two examples show how the sensitivity of the system is beyond imagination.

It can be seen from the two graphs above that both products grow very fast intially. Most of the right-side figure is covered by the last big jump, and the growth of the previous steps is almost invisible. This method is not feasible for a larger system since hundreds of iterations might be needed, and our present-day computers are not powerful enough to handle the computations. We need to find a new way.

4. **Chen's model.** In 1957, Hua gave some lectures in the Institute of Mathematics of the Chinese Academy of Sciences on the input-output method, popular in the economic community at that time, established by Wassily Leontief in 1936 (Nobel Prize winner in Economics in 1973). Leontief removed the traditional basic requirement of distinguishing between production data and consumption data, simplified the conventional model, and successfully applied the new model in practice. From the Perron-Frobenius Theorem 2.2, Hua realized that production data and consumption data should be clearly differentiated. In 1958–1959, in the textbook "Introduction to Advanced Mathematics" (Volume 4) written for students of the Mathematics Department at the University of Science and Technology of China, he gave a new proof of the aforementioned theorem. The core part of Hua's model is keeping the tradition of clearly separating production data and consumption data. For models involving consumption, new methods are needed. In fact, this was a major challenge in our input-output model. Between 1984 and 1985, Hua tried several methods and published more than 10 research reports in the "Chinese Science

Bulletin". The key point was that each year, a portion of the increase in production should be allocated for consumption. Hua proposed two approaches for this. In papers [9] and [8], he used

$$x_n - \xi_n = x_{n+1}A, \tag{5}$$

where ξ_n is the consumption; in the paper [10], he denoted the left-hand side of the equation above by y_n, and hence (5) becomes $y_n = x_{n+1}A$. Suppose that, for some $\alpha \in (0,1)$, an α portion of the increment in production in the n-th year, i.e., $\alpha(x_n - y_{n-1})$, is used for consumption, then the amount that d-vector can be used for reproduction in the $(n+1)$-th year is

$$\begin{aligned} y_n &= x_n - \alpha(x_n - y_{n-1}) \\ &= (1-\alpha)x_n + \alpha y_{n-1} \\ &= (1-\alpha)y_{n-1}A^{-1} + \alpha y_{n-1} \\ &= y_{n-1}[(1-\alpha)A^{-1} + \alpha I]. \end{aligned}$$

Thus we obtain the model

$$y_n = y_{n-1}B,\ n \geqslant 1, \tag{6}$$

where the matrix $B = (1-\alpha)A^{-1} + \alpha I$ is no longer non negative and $\alpha \in (0,1)$ is the consumption ratio. In this case, the economic growth rate is $(1-\alpha)\rho(A)^{-1} + \alpha$.

As far as we know, Model (6) was in use until around October of 2021. However, Hua was not satisfied with any of the aforementioned models. About 50 days before his death, he removed all previous research reports from the references of his manuscript [11] "Mathematical Theory of Large-Scale Optimization in Planned Economy" and proposed a new model, that is, in (5), set

$$\xi_n = \gamma(x_{n+1} - x_n),$$

where $\gamma \in (0,1)$ is the consumption ratio, and so

$$x_n = x_{n+1}A + \gamma(x_{n+1} - x_n) = x_{n+1}(A + \gamma I) - \gamma x_n.$$

Hence, we have

$$x_n = x_{n+1}A_\gamma, \quad A_\gamma = \frac{A + \gamma I}{1+\gamma}, \quad \rho(A_\gamma) = \frac{\rho(A) + \gamma}{1+\gamma}.$$

This returns to the model with no consumption and with non-negative structure matrix

$$x_0 = x_n A_\gamma^n. \tag{7}$$

Unfortunately, Model (7), Hua's "Last Thought", remained dormant for 37 years before it was awakened by Mu-Fa Chen [4].

The development rate of the economic system corresponding to Model (7) is $1/\rho(A_\gamma)$ when $\rho(A) \geqslant 1$, the economic system is abnormal and further research is required. Thus we assume $\rho(A) < 1$. Then the corresponding growth rate is

$$\frac{1}{\rho(A_\gamma)} - 1 = \frac{1-\rho(A)}{\gamma + \rho(A)} \downarrow \frac{1-\rho(A)}{1+\rho(A)} > 0,\ \gamma \uparrow 1.$$

This shows that, for any $\gamma \in (0,1)$, Model (7) always has a positive growth rate, which is inconsistent with reality. Therefore, Model (7) has a flaw. Noting that

$$\frac{1-\rho(A)}{\gamma+\rho(A)} \downarrow 0 \iff \gamma \uparrow \infty,$$

Chen proposed the following model. Take

$$\gamma = \frac{\alpha}{1-\alpha},$$

where $\alpha \in (0,1)$ is the consumption parameter, and so $\gamma \in (0,\infty)$. This is completely different from the assumption $\gamma \in (0,1)$ in Model (7). In practice, $\gamma > 1$ may occur. This results in the new model

$$x_0 = x_n A_\alpha^n, \qquad n \geqslant 1, \tag{8}$$

where

$$A_\alpha = (1-\alpha)A + \alpha I, \ \rho(A_\alpha) = (1-\alpha)\rho(A) + \alpha.$$

Obviously A_α ($\alpha < 1$) has the same maximal left and right eigenvectors u and v as A. Model (8) is called **Chen's model**.

The matrix A_α in Chen's model can be constructed in two steps:

$$A \overset{\text{(i)}}{\Longrightarrow} (1-\alpha)A \overset{\text{(ii)}}{\Longrightarrow} (1-\alpha)A + \alpha I = A_\alpha.$$

(i) Since a portion of the resources is used for consumption, α times of the structure matrix A is deducted from A. Thus, only $1-\alpha$ times of A is actually invested in reproduction.

(ii) The deducted part is not used for reproduction, so the off-diagonal elements of the deducted part of the new structure matrix are all 0; each product still contributes an α portion to the new structural matrix, to be used for consumption.

5. **Key transform of non negative matrices: Chen's transform.** As mentioned before, when the structure matrix A is a transition probability matrix, Chen has given a proof of Hua's fundamental theorem. How do we transform a general structure matrix into a transition probability matrix? Chen has been working on this topic for a long time.

The first step is to figure out what needs to be done to transform A into a transition probability matrix P. We know that, for transition probability matrix P, its maximal left eigenvector π is the stationary distribution of P. Markov chain theory tells us that, under the irreducibility condition, the stationary distribution of P is unique.

Definition 5.1. The nonnegative matrix having sum 1 of each row is called a transition probability matrix.

Lemma 5.2. *The maximal eigenvalue of a transition probability matrix equals one.*

Proof. Let $\|x\|_\infty = \sup_k |x^{(k)}|$, $\|P\|_\infty = \sup_{x\neq 0} |Px|_\infty / \|x\|_\infty$. Obviously, 1 is an eigenvalue of P, and its corresponding eigenvector is $\mathbb{1}$; we can get that its maximal eigenvalue has a lower bound of 1. On the other hand, according to the definition of the eigen-equation:

$$Px = \lambda x.$$

Because

$$(Px)(k) = \sum_j p_{kj} x^{(j)} \leqslant \|x\|_\infty \sum_j p_{kj} = \|x\|_\infty,$$

hence $\|Px\|_\infty| \leqslant \|x\|_\infty$. We get the maximal eigenvalue of P

$$\lambda_{\max} = \sup_{x \neq 0} \frac{\|Px\|_\infty}{\|x\|_\infty} \leqslant 1.$$

□

Remark 5.3. If P is in addition irreducible, then the conclusion of Lemma 5.2 can be derived directly from the C-W formula. In fact, since 1 is the eigenvalue of P, and $\mathbb{1}$ is the corresponding eigenvector, by applying the C-W formula to the vector $\mathbb{1}$ we can get $\lambda_{\max} \leqslant 1$.

For a positive vector w, we use D_w to denote the diagonal matrix with diagonal entries given by the components of w. For any vector x, we define the component product $w \odot x = (w^{(1)}x^{(1)}, \cdots, w^{(d)}x^{(d)})$ and

$$A_w \hat{=} D_w^{-1} \frac{A}{\rho(A)} D_w. \tag{9}$$

Then we have the following result.

Theorem 5.4 (Chen 1989, 1992, 2022)**.** *From (9), the following conclusions hold:*
(i) *A_w is a transition probability matrix P if and only if $w = v$;*
(ii) *The maximal left eigenvector of P is $\mu := u \odot v$;*
(iii) *$\pi := u \odot v/(uv)$ is a stationary distribution of P: $\pi = \pi P = \pi P^n$.*

Proof. Without loss of generality, we assume $\rho(A) = 1$.
(i) From (9), we know that

$$A_w \mathbb{1} = D_w^{-1} A D_w \mathbb{1} = D_w^{-1} A w \stackrel{?}{=} \mathbb{1}.$$

Conclusion (i) holds if and only if $Aw = D_w \mathbb{1}$, i.e., $Aw = w$. Therefore, A_w is a transition probability matrix if and only if $w = v$.
(ii) It follows from (i) that

$$P = D_v^{-1} \frac{A}{\rho(A)} D_v. \tag{10}$$

Thus we have

$$\mu P = u \odot vP = u \odot v D_v^{-1} A D_v = uAD_v = uD_v = u \odot v = \mu,$$

That is, $u \odot v$ is the maximal left eigenvector of P.
(iii) Define $\pi = u \odot v/(uv)$. It is easy to verify that π is the stationary distribution of P. Combining

$$P\mathbb{1} = D_v^{-1} A D_v \mathbb{1} = D_v^{-1} A v = D_v^{-1} v = \mathbb{1}$$

with Lemma 5.2, we get that $\rho(P) = 1$ and the maximal right eigenvector of P is $\mathbb{1}$.
□

Remark 5.5. (1) The sufficiency part of Theorem 5.4(i) was obtained by Chen in 1989 and published in 1992. The necessity part was proved in 2022. Theorem 5.4 (ii) and (iii) were also proved by Chen in 2022.
(2) We will call (10) **Chen's transform** of A, which was given by Chen in 2022.

The equilibrium $\mu = u \odot v$ plays an important role in the present theory, as shown in the next section.

Now we present Chen's proof of Hua's fundamental theorem in the case of general structure matrices.

Proof of Hua's fundamental theorem. It follows from Theorem 5.4 (i) that

$$D_v^{-1}\left(\frac{A}{\rho(A)}\right)^n D_v = P^n. \tag{11}$$

Thus

$$\left(\frac{A}{\rho(A)}\right)^n = D_v P^n D_v^{-1}.$$

Assume that $\{x_n\}_{n\geqslant 0}$ satisfies $(x_0 D_v)\mathbb{1} = 1, x_0 = x_n A^n$ and $x_n \geqslant 0$ $(n \geqslant 0)$. By the definitions of u and v and the assumption $(x_0 D_v)\mathbb{1} = 1$, we know that, up to a multiplicative constant, $x_0 = u$. Note that

$$x_0 D_v = x_n A^n D_v = [\rho(A)^n x_n D_v]\, D_v^{-1}\left(\frac{A}{\rho(A)}\right)^n D_v = [\rho(A)^n x_n D_v]\, P^n. \tag{12}$$

Denote $y_n = \rho(A)^n x_n D_v$, and (12) can be rewritten as

$$y_0 = y_n P^n, n \geqslant 0$$

and $y_0\mathbb{1} = 1$ $(n \geqslant 0)$. Combining Theorem 5.4 and the proof of Hua's fundamental theorem in the special case of $A = P$, we can get $y_0 = u \odot v/(uv)$, that is, $x_0 = u/(uv)$. In order to make all x_n non-negative, the initial value must be u. □

For a given P and μ_0, define $\mu_0 = \mu_n P^n$, $n \geqslant 1$. $\{\mu_n\}_{n\geqslant 0}$ is called the iterative sequence of μ_0 under P. Similarly, the iterative sequence $\{x_n\}_{n\geqslant 0}$ of x_0 under A is defined by $x_0 = x_n A^n$, $n \geqslant 1$. The following result shows the equivalence of these two iterative sequences.

Theorem 5.6. *$\{\mu_n\}_{n\geqslant 0}$, $\{x_n\}_{n\geqslant 0}$, and v satisfy the following identities:*

$$\mu_n = \rho(A)^n x_n \odot v,\ n \geqslant 0, \tag{13}$$

$$x_n = \rho(A)^{-n}\mu_n \odot v^{-1},\ n \geqslant 0. \tag{14}$$

Proof. Let $\{x_n\}_{n\geqslant 0}$ be the iterative sequence of x_0 under A. Then (12) holds. It follows from $x_0 D_v = \rho(A)^0 x_0 \odot v$ that $\mu_n := \rho(A)^n x_n \odot v (n \geqslant 0)$, satisfying $\mu_0 = \mu_n P^n, n \geqslant 1$. That is, (13) holds. (14) can be proved similarly. □

Multiplying both sides of (13) by the vector $\mathbb{1}$ on the right side yields $\rho(A)^n x_n v = \mu_n\mathbb{1}$. In particular, if we take $x_0 = u$, then $\mu_0 = u \odot v$, the normalization condition is $uv = \mu_0\mathbb{1} = 1 = \pi\mathbb{1}$. Note that μ_n and x_n are related like in (13).

Next we consider Chen's transform P_α of the structure matrix $A_\alpha = (1-\alpha)A+\alpha I$ in the case with consumption. Obviously, the maximum eigenvalue of A_α is $\rho(A_\alpha) = (1-\alpha)\rho(A) + \alpha$. Note that A_α and A have the same maximal right eigenvector v, thus Chen's transform of A_α is

$$\begin{aligned} P_\alpha =& D_v^{-1}\frac{A_\alpha}{\rho(A_\alpha)}D_v \\ =& \rho(A_\alpha)^{-1}\left[(1-\alpha)D_{v^{-1}}AD_v + \alpha I\right] \\ =& \rho(A_\alpha)^{-1}\left[(1-\alpha)\rho(A)D_v^{-1}\frac{A}{\rho(A)}D_v + \alpha I\right] \\ =& (1-\beta_\alpha)P + \beta_\alpha I, \end{aligned} \tag{15}$$

where $\beta_\alpha = \alpha/\rho(A_\alpha)$.

(15) shows that P_α ($\alpha<1$) has the same maximal left and right eigenvectors $u\odot v$ and $\mathbb{1}$. One can similarly prove the equivalence of the two iterative algorithms for A_α and P_α.

To conclude this section, let us illustrate the power of Chen's transform. For this, we return to the stability testing of Examples 3.4 and 3.5, using P instead of A. The results are given by Figures 3 and 4 below. Certainly, the initials for P should be different from those of A.

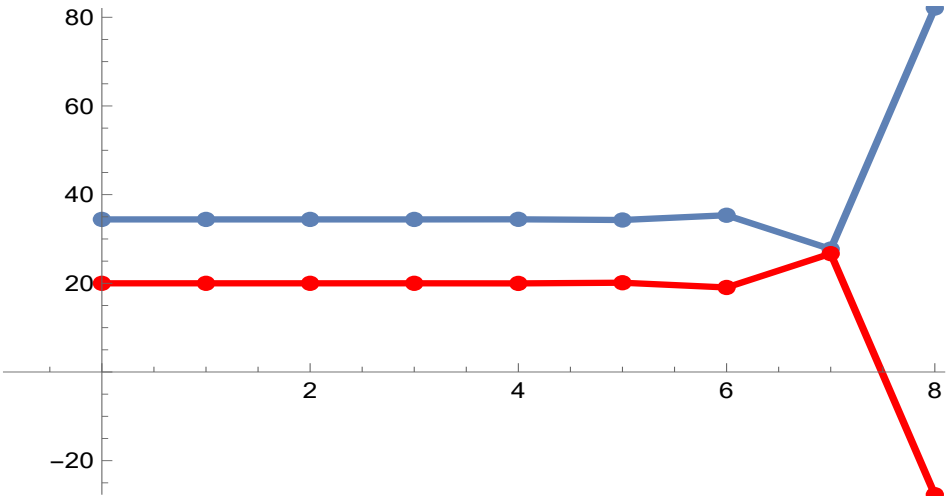


FIGURE 3. New figure corresponding to Figure 1. Initial $(34.41181135,\ 20)$, $T = 8$.

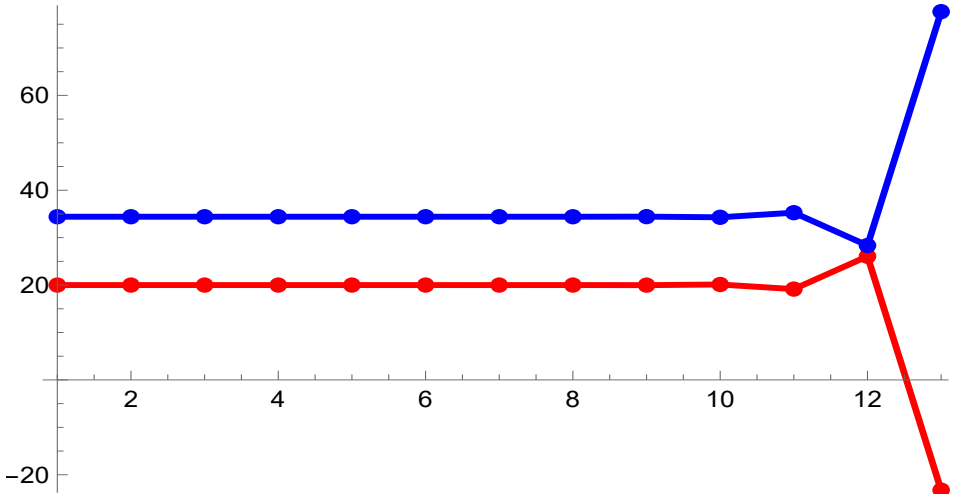


FIGURE 4. New figure corresponding to Figure 2. Initial $(34.41179182,\ 20)$, $T = 13$.

Unexpectedly, the stability for P and A are completely the same (i.e., having the same collapse time and the same collapse place (at the same product)). Surprisingly, the picture generated by P is much better than the one generated by A. Therefore, we use P only in practice for stability testing.

6. **Product ranking and classification.** The second part of Theorem 3.2 is Hua's main contribution in economic optimization, and as far as we know, it has never appeared before. It says that, if $x_0 \neq u$, then the system will collapse at an exponential rate. Therefore, it is important to understand the classification of products in the economic system: pillar products, intermediate products, and weak products, because the system often collapses at some weak products.

For a non negative and irreducible matrix A, the maximal left eigenvector u contains only two characteristics of A. But the maximal left eigenvector $\mu = u \odot v$ of P obtained from A via Chen's transform contains all three characteristics of A. As mentioned before, $u \odot v$ represents the true total value of the products and has a unified dimension. Therefore, compared with using the maximal left eigenvector u of A, it is more scientifically accurate to use the maximal left eigenvector μ of P to rank products. Furthermore, as we have seen from the last section, using P or A results in completely the same stability, but the amplitude of oscillation of the former is less than that of the latter.

To illustrate the significance of product ranking, we give some practical examples. Figure 5 gives the rankings of 42 products in China's input-output tables in 2017 (red), 2012 (blue), and 2007 (black). The National Bureau of Statistics in China compiles an input-output table every 5 years, so Figure 5 contains detailed information about China's economic situation in a 15 year period. For details, see Chapter 4 of [7]. For ease of observation, the vertical axis represents multiples of the equilibrium solution of the transition probability matrix P. The three curves are surprisingly similar. Before we analyze the figure, we should inform our readers that, in the 2007 input-output table, we did a numerical interpolation for the missing value of the 24th product "Metal Products, Machinery and Equipment Repair Services" in order to get a complete curve of 42 products. The main differences among the blue, black, and red curves are that the top product in the red curve is the 20th product, "Communication Equipment, Computers and other Electronic Equipment", while the top product in the blue cure is the 12th product, "Chemical Products". It is not difficult to understand why such a change occurred between 2012 and 2017: this period is when mobile phones and other electronic devices started to become popular. If we take the blue curve of 2012 as the benchmark and mark the top 6 products in a decreasing order with circled numbers, the top 6 of the blue curve and that of the black curve are basically the same except for the 5th one.

One can also see from Figure 5 that the rankings of the 30th product "Transportation", the 33th "Finance", the 34th "Real Estate", and the 35th "Leasing and Business" in these three curves gradually increase. We also see that, contrary to traditional popular belief, finance and real estate are not pillar products.

Figure 6 is the distribution function diagram generated by π. Arranging the components of π in ascending order $p_1 < p_2 < \cdots < p_{42}$, we get the discrete cumulative distribution function as follows: $p_1, p_1 + p_2, \cdots, \sum_{k=1}^{42} p_k$.

We plot all three cumulative distribution graphs in Figure 6. There are two graphs in Figure 6: the upper graph is the full graph, and the lower graph is a

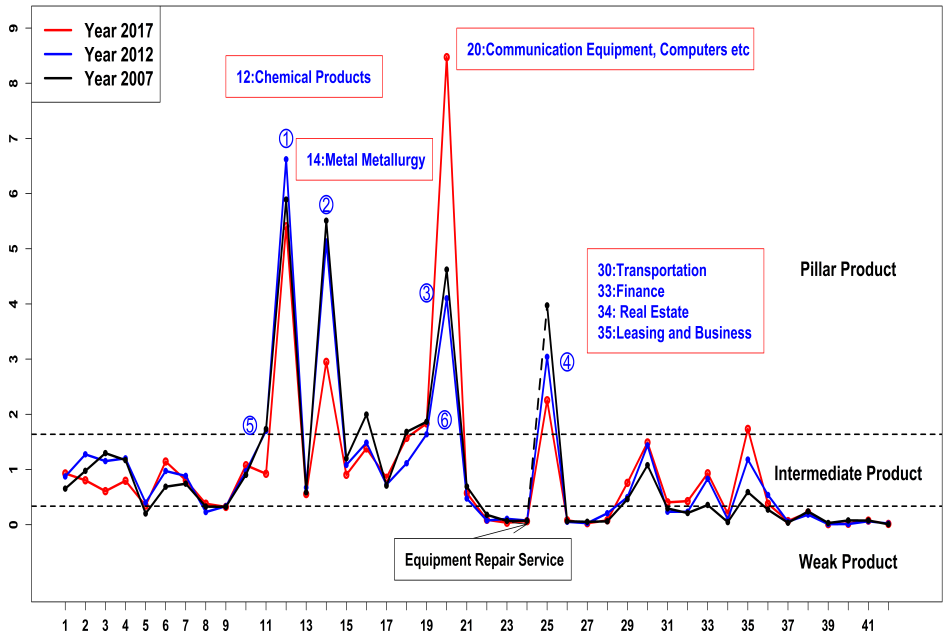


FIGURE 5. The equilibrium diagrams of Year 2017's 42 products, Year 2012's 42 products, and Year 2007's 41 products.

partial one. The black, blue, and red curves are the cumulative distribution curves of 2007, 2012, and 2017, respectively. The horizontal axis in Figure 6 is the rank order of each product, and the vertical axis is the cumulative distribution of the first n products. To the left of the left vertical line, the cumulative distribution is less than or equal to 5%, and to the right of the right vertical line, the cumulative distribution is larger than or equal to 50%. Products to the left of the left vertical line are the weak products. Products to the right of the right vertical line are the pillar products. Products between the two vertical lines are the intermediate products. One can see from Figure 6 that there are 17 weak products and 5 pillar products in 2007. One can also see that the cumulative distribution curves spanning 15 years are very consistent, indicating that using cumulative distribution to classify product grades is reliable. Since the number of weak products is relatively large, we further divided them in the lower graph. One can see that there 8 or 9 products with cumulative distribution less than or equal to 1%.

Figure 5 and 6 are taken from [13].

7. **Quantum wave probability.** The materials of this section are mainly taken from Chen's paper [5]. Consider the complex matrix A. There are some extended versions of the Perron-Frobenius theorem, but the results are quite limited. However, a point in Section 5, the transform from A to P, still makes sense.

Definition 7.1. A complex matrix A is called an SR1 matrix if $A\mathbb{1} = \mathbb{1}$, i.e., the row sums are all equal to 1.

Assume $\lambda \neq 0$ is a simple eigenvalue of the complex matrix A. Similar to Chen's theorem on the key transform in Section 5, we have the following result.

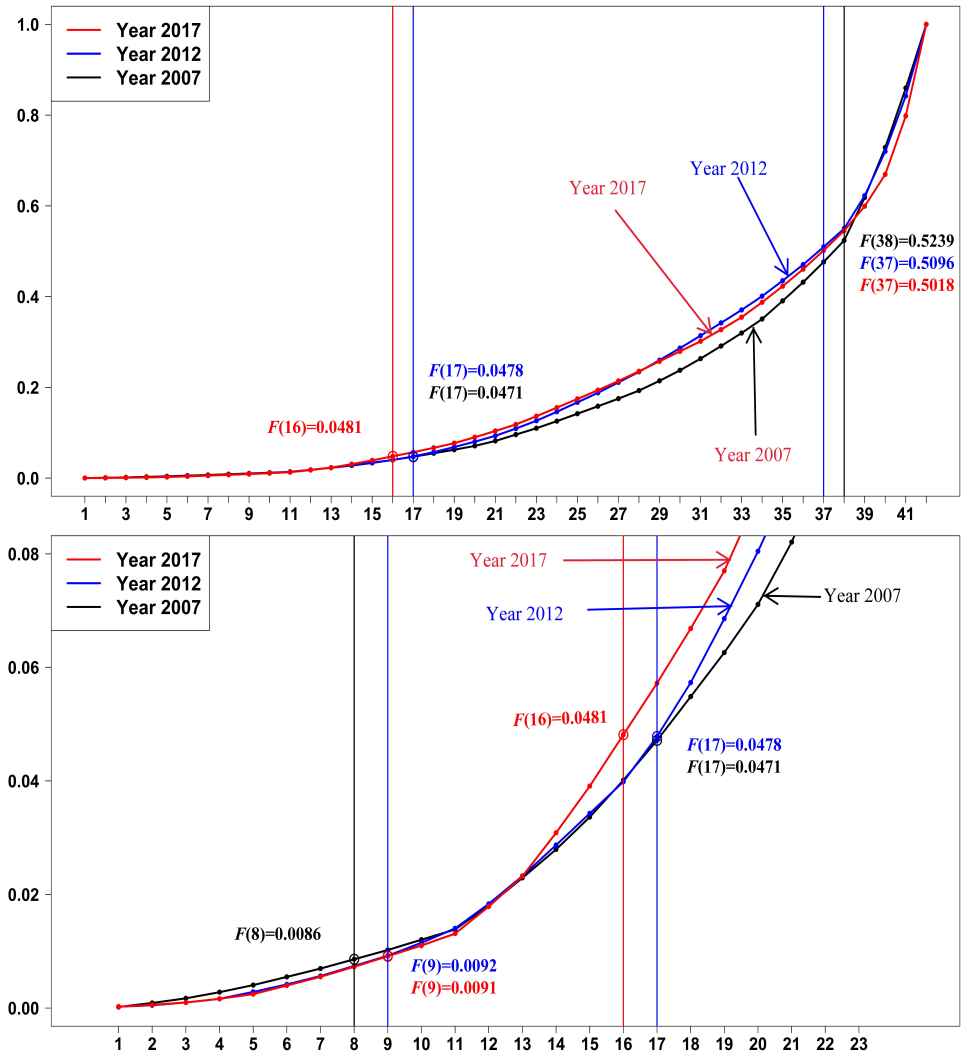


FIGURE 6. Cumulative distribution of equilibrium solutions of 42 products in 2017 and 2012.

Theorem 7.2 (Generalized Chen's Theorem)**.** *Suppose that* (λ, u) *and* (λ, v) *are left and right eigenpairs of the complex matrix* A*, respectively,* $uA = \lambda u$*,* $Av = \lambda v$*, and all components of* v *are non-zero. For a column vector* w *with all its components being non-zero, define*

$$R_w = D_w^{-1} \frac{A}{\lambda} D_w.$$

Then the following conclusions hold:

(1) R_w *is an SR1 matrix if and only if* $w = v$*;*

(2) R_v *has a left eigenpair* $(\lambda, u \odot v)$*:* $(u \odot v)R_v = \lambda(u \odot v)$.

For a Hermitian matrix A, left and right eigenvectors of A corresponding to the same eigenvalue are conjugates of each other, so $u = \bar{v}$. Thus by Theorem 7.2, the left-eigenvector of R_v is $u \odot v = \bar{v} \odot v$. Hence, we have the following corollary.

Corollary 7.3. *If in the theorem above we further assume that A is Hermitian, then the left eigenvector of R_v is $\bar{v} \odot v$.*

We know that a discretized Schrödinger operator corresponds to a Hermitian matrix, and its eigenvalues correspond to discrete energy levels. In quantum mechanics, the square of the modulus of the wave function is the probability density of particle distribution, that is:

$$\pi = \frac{\bar{v} \odot v}{|v|^2} = \frac{\left(|v_1|^2, |v_2|^2, \cdots, |v_d|^2\right)}{|v|^2}.$$

In other words, the vector $\bar{v} \odot v$ contains three characteristics of the Hermitian matrix A: the maximal eigenvalue λ and the corresponding left and right eigenvectors u and v. Since $\bar{v} \odot v$ represents the square of the modulus of A's wave function (equivalently, the eigenvector) v, this explains why in Born's comment about matrix mechanics one should use "the square of the modulus" instead of "the modulus". Once $|v|<\infty$, one can use its normalized probability measure π instead of $\bar{v} \odot v$ to denote the probability density of particle distribution. In summary, this is just an explanation of the same thing in two different languages, and so there is no objective randomness; just as Einstein said, "God does not play dice".

Note that the real symmetric matrix is a special case of the complex Hermitian matrix, so the above conclusions can also be applied to the principal component analysis.

8. **Economic forecasting and adjustment.** In practice, for a model with consumption, to determine a suitable economic growth rate, one needs to go through a number of adjustments and computations. For a predetermined growth rate, we first compute the consumption parameters and then the available consumption. The program goes as follows:

$$\text{Growth rate} \Rightarrow \text{Consumption parameters}$$
$$\Rightarrow \text{Available consumption} \overset{?}{\geqslant} \text{Planned consumption}$$

If "$\geqslant$" holds, then the system can achieve the predetermined speed, and one can stop. Otherwise, one can test again by reducing the growth rate. One can use the optimization method to design these tests.

Theorem 8.1 ([4]). *If the economic growth rate satisfies $0 < \delta < \min\{1, \rho(A)^{-1} - 1\}$, then the available consumption in the $(n+1)$-th year is*

$$\xi_n = \frac{1 - (1+\delta)\rho(A)}{\delta}(x_{n+1} - x_n), \tag{16}$$

where $\{x_n\}$ is the iterative sequence with given initial value x_0 under A_α in Chen's model:

$$x_n = x_{n+1}A_\alpha, \qquad \alpha := \frac{(1+\delta)^{-1} - \rho(A)}{1 - \rho(A)}.$$

Conversely, one can use the fact the consumption in the $(n+1)$-th year does not exceed ξ_n to determine the maximum growth rate δ.

Proof. According to Theorem 3.2 (i), the economic growth rate (denoted by δ) is $1/\rho(A_\alpha)-1$, so

$$\delta=\frac{1}{\rho(A_\alpha)}-1=\frac{1}{(1-\alpha)\rho(A)+\alpha}-1. \tag{17}$$

Solving the equation (17), we get the consumption parameter

$$\alpha=\frac{(1+\delta)^{-1}-\rho(A)}{1-\rho(A)}=:\alpha(\delta) \tag{18}$$

and the consumption multiple

$$\gamma_\alpha=\frac{\alpha}{1-\alpha}=\frac{(1+\delta)^{-1}-\rho(A)}{1-(1+\delta)^{-1}}=:\gamma(\delta). \tag{19}$$

Therefore, the available consumption in the $(n+1)$-th year is

$$\xi_n(\alpha)=\gamma_\alpha(x_{n+1}-x_n)=\frac{(1+\delta)^{-1}-\rho(A)}{1-(1+\delta)^{-1}}(x_{n+1}-x_n).$$

That is, (16) holds. □

Combining (18) and (19) with $\rho(A)<1$, we get that $\alpha(\delta)$ and $\gamma(\delta)$ are both monotonically decreasing functions of δ. This implies that the consumption decreases when the economic growth rate increases, and consumption increases when the economic growth rate decreases.

It can be seen from the proof above that, if the economic growth rate δ is given, we can get α from (18) and then A_α. Regarding the d-vector x_n as input, one can use Chen's model $x_n=x_{n+1}A_\alpha$ to find the output d-vector x_{n+1} of the $(n+1)$-th year, and then get the available consumption of the $(n+1)$-th year.

Conversely, if the available consumption of the $(n+1)$-th year and the input d-vector x_n are given, the core of predicting the economic growth rate δ lies in solving the consumption parameter α. For this, we have the following result.

Corollary 8.2. *If the actual required consumption (planned consumption) is $\bar{\xi}_n$, then we only need to find α so that $\bar{\xi}_n\leqslant\xi_n(\alpha)$. Naturally, we define $\bar{\alpha}=\inf\{\alpha\in(0,1):\xi_n(\alpha)\geqslant\bar{\xi}_n\}$. $\bar{\alpha}$ is the consumption parameter corresponding to $\bar{\xi}_n$. Since $\xi_n(\alpha)$ increases with α, we have, for any $\alpha\geqslant\bar{\alpha}$,*

$$\xi_n(\alpha)\geqslant\xi_n(\bar{\alpha})\geqslant\bar{\xi}_n.$$

Then the planned consumption $\bar{\xi}_n$ is alway less than or equal to the available consumption $\xi_n(\alpha)$. $\delta(\bar{\alpha})$ can be determined using the definition of $\bar{\alpha}$ and (17).

Given x_n and A, the available consumption $\xi_n(\alpha)=\gamma_\alpha(x_{n+1}-x_n)$ can be uniquely determined by $\alpha\in(0,1)$. In fact, $\xi_n(\alpha)$ is proportional to $x_{n+1}-x_n$. More precisely, the ratio of these two is $\alpha/(1-\alpha)$; this is a carefully designed special quantity. In general, one cannot expect that an arbitrary planned consumption $\bar{\xi}_n$ will be equal to the available consumption $\xi_n(\alpha)$ for a certain α. However, such a strong restriction is not really needed in practice, because it is not always necessary to reduce economic growth to increase consumption. One can adjust the consumption structure or imports to achieve this.

9. **Product regulation and optimization of economic structure.** By "Product regulation" we mean to determine the products or industries that should be prioritized (or eliminated) in order to adjust and optimize the economic structure. This should be based on the current economic situation and needs rigorous computation and analysis.

This is a very difficult problem. Chen raised this as an open problem in 2005 (see [2]). It was not until 2022 that he completely solved this problem. The biggest challenge is to find the optimization criteria. In traditional optimization theory, one tries to find the values of certain variables (under certain constraints) so that the chosen cost function attains the optimum. The problem is that, in our economic model, we do not know what the cost function is.

In our economic model here, we have actually done two optimizations: one is to achieve the fastest growth rate and the other is make sure collapse never happens. Hua's fundamental theorem tells us that the unique solution of both optimizations is the maximal left-eigenvector u of A. This prompts us to take the d-vector $\tilde{u} = (\tilde{u}_k)$ as the target. In other words, first, we should find a target $\tilde{u}$ for product regulation. Next, we construct a new structure matrix $\tilde{A}$ with $\tilde{u}$ being its maximal left eigenvector. This is a very complicated problem. For example, if we consider a system with 100 products, we need to determine the 10^4 elements of the new structure matrix. However, the given new equilibrium only provides 100 data points. Where can we find the remaining 9900 data points? In order to fill in the missing information of the new matrix, we can require the new structure matrix $\tilde{A}$ to be as close as possible to the original structure matrix A.

To complete the construction of the new structure matrix, we first make some necessary preparations in the next section, and then we return to the topic in Section 11.

10. **Markov chain, dual Markov chain, and invariants of economic structure matrix.** Recall that Chen's transform $\tilde{A}$ of the matrix A is defined by

$$\frac{\tilde{A}}{\rho(\tilde{A})} = D_w^{-1}\frac{A}{\rho(A)}D_w, \qquad w > 0. \tag{20}$$

It follows from Theorem 5.4(i) that $\tilde{A}$ is a transition probability matrix P if and only if w is the maximal right eigenvector v of A. We will often call P a **Markov chain**. The **three major characteristics** (the maximal eigenvalue and the corresponding maximal left and right eigenvectors) of P are $(1, u \odot v, \mathbb{1})$.

Let A^* be the transpose of A and u^* the corresponding maximal right eigenvector of A^*. It follows from (20) that

$$D_w^{-1}\frac{A^*}{\rho(A)}D_w, \qquad w > 0, \tag{21}$$

gives another transition probability matrix if and only if $w = u^*$. The three characteristics of (21) are $(1, u^* \odot v^*, \mathbb{1})$ (of course, $u^* \odot v^* = u \odot v$). To remove $*$ in the above formula, we need to make the transposition and then get

$$Q_w := D_w\frac{A}{\rho(A)}D_w^{-1}.$$

Now the column sums of Q_w are all equal to 1 if and only if $w = u$. We call Q_u the **Dual Transition Probability Matrix** or **Dual Markov chain** (abbrev.**Dual Chain**). Its three major characteristics are $(1, \mathbb{1}^*, u \odot v)$.

Before studying the invariants of the economic structure matrix, we first review the simplest invariant:

$$\pi = \frac{\text{circumference of the circle}}{2\,r} = \frac{\text{area of the circle}}{r^2},$$

which omits the basic parameter describing the size of a circle–radius r. Recall our consumption model:

$$A_\alpha = (1-\alpha)A + \alpha I, \qquad \alpha \in [0,1).$$

Omitting the spectral radius $\rho(A_\alpha)$ that describes the development speed, this family of matrices has two invariants; they have the same maximal left and right eigenvectors u and v. Two of the three major characteristics become invariants (independent of α). We have seen that these two invariants (vectors rather than scalars) play an important role in economic theory.

Now for a given P, we consider the inverse transform of the transform (20). For any positive w, we can define Chen's transform of P

$$A_w := D_w P D_w^{-1}.$$

Since A_w is similar to P, we have $\rho(A_w) = 1$. It is easy to show that w is the maximal right eigenvector of A_w. Therefore, by (20), the transition probability matrix derived from A_w is P. This shows that the family of square matrices

$$\mathscr{A}_P = \{A_w = D_w P D_w^{-1} : w > 0\}$$

has the same **invariant square matrix** P, which is a square matrix instead of a vector.

We can deal with the dual case similarly. For a fixed dual Markov chain Q, we define the family of square matrices

$$\mathscr{A}_Q := \{A_w = D_w^{-1} Q D_w : w > 0\}.$$

For any $A_w \in \mathscr{A}_Q$, it is obvious that $\rho(A_w) = \rho(Q) = 1$. It is easy to show that w is the maximal left eigenvector of A_w, so the dual Markov chain derived from A_w coincides with Q. This means that the family of square matrices $\mathscr{A}_Q$ has Q as an **invariant** (**invariant square matrices**).

The above invariant vectors or invariant matrices are collectively called **Chen's invariants**.

Note that since $w > 0$, we can replace w by w^{-1}, if necessary. In the definition of the set $\mathscr{A}_Q$, $D_w^{-1} Q D_w$ becomes $D_w^{\mp} Q D_w^{\pm}$, so the set $\mathscr{A}_Q$ is large enough. This is useful in practice because we often need to adjust $w = \tilde{u}$. Of course, in the definition of $\mathscr{A}_Q$, we can replace A_w by $A_w/\rho(A_w)$ to add a degree of freedom to $\mathscr{A}_Q$.

11. **Product regulation and optimization of economic structure (continued).** Now, we return to the topic of Section 9: construct a new structure matrix $\tilde{A}$ with $\tilde{u}$ as the equilibrium solution, making it as close as possible to the original matrix A. We are given A and its maximal left eigenvector u, and the target equilibrium solution $\tilde{u}$. From the previous section, we have a completely determined dual chain

$$Q_u := D_u \frac{A}{\rho(A)} D_u^{-1}.$$

It is in a one-to-one correspondence with $A/\rho(A)$:

$$\frac{A}{\rho(A)} = D_u^{-1} Q_u D_u.$$

Replacing A with the desired $\tilde{A}$, we formally get two equations parallel to the equations above

$$Q_{\tilde{u}} := D_{\tilde{u}} \frac{\tilde{A}}{\rho(\tilde{A})} D_{\tilde{u}}^{-1}, \qquad \frac{\tilde{A}}{\rho(\tilde{A})} = D_{\tilde{u}}^{-1} Q_{\tilde{u}} D_{\tilde{u}}.$$

Since $\tilde{u}$ is given, $\tilde{A}$ and the corresponding $Q_{\tilde{u}}$ are the only unkowns in the last two equations. To make $\tilde{A}$ as close to A as possible, one should require their invariants to be as close as possible. The simplest way to do this is to let $Q_{\tilde{u}} = Q_u$. Starting from $Q_{\tilde{u}}$ and Q_u, we can define, as we did in the previous section, two families of square matrices with $Q_{\tilde{u}}$ and Q_u as invariants:

$$\mathscr{A}_{Q_u} = \left\{ A : \frac{A}{\rho(A)} = D_w Q_u D_w^{-1},\ w > 0 \right\},$$
$$\mathscr{A}_{Q_{\tilde{u}}} = \left\{ \tilde{A} : \frac{\tilde{A}}{\rho(\tilde{A})} = D_w Q_{\tilde{u}} D_w^{-1},\ w > 0 \right\}.$$

When $Q_{\tilde{u}} = Q_u$, $\mathscr{A}_{Q_{\tilde{u}}} = \mathscr{A}_{Q_u}$. Thus we can find the only unknown quantity $\tilde{A}/\rho(\tilde{A})$ from the above system of equations:

$$\begin{aligned}
\frac{\tilde{A}}{\rho(\tilde{A})} &= D_{\tilde{u}}^{-1} Q_{\tilde{u}} D_{\tilde{u}} \quad (\text{first take } w = \text{the known } \tilde{u} \text{ in } \mathscr{A}_{Q_{\tilde{u}}}) \\
&= D_{\tilde{u}}^{-1} Q_u D_{\tilde{u}} \quad (\text{let } Q_{\tilde{u}} = \text{the known } Q_u) \\
&= D_{\tilde{u}}^{-1} D_u \frac{A}{\rho(A)} D_u^{-1} D_{\tilde{u}} \quad (\text{plug in the explicit expression of } Q_u) \\
&= D_{\tilde{u}^{-1} \odot u} \frac{A}{\rho(A)} D_{u^{-1} \odot \tilde{u}} \quad (\text{simplify the right-hand side}) \\
&= D_{\tilde{u} \odot u^{-1}}^{-1} \frac{A}{\rho(A)} D_{\tilde{u} \odot u^{-1}}.
\end{aligned}$$

Because $\{A_\alpha, \alpha \in [0,1)\}$ have the invariants u and v, this result can be immediately extended to the case with consumption, which leads to the following theorem.

Theorem 11.1. (*Chen's Economic Structure Optimization Theorem*) *The optimal matrix $\tilde{A}_\alpha$, with the target $\tilde{u}$, of the structure matrices A_α with the maximal left eigenpair $(\rho(A_\alpha), u)$ can be determined by* (20) *and $w = \tilde{u} \odot u^{-1}$:*

$$\frac{\tilde{A}_\alpha}{\rho(\tilde{A}_\alpha)} = D_{\tilde{u} \odot u^{-1}}^{-1} \frac{A_\alpha}{\rho(A_\alpha)} D_{\tilde{u} \odot u^{-1}}, \qquad \alpha \in [0,1). \tag{22}$$

Furthermore, the maximal left and right eigenvectors of $\tilde{A}_\alpha$ are $\tilde{u}$ and $\tilde{v} = v \odot u \odot \tilde{u}^{-1}$.

Proof. We only need to show the last assertion. Since $\{A_\alpha : \alpha \in [0,1)\}$ have the same maximal left and right eigenvectors u and v, we will omit α in this proof.

(a) First, we show that $\tilde{u}$ is the maximal left eigenvector of $\tilde{A}$, that is,

$$\tilde{u} \frac{\tilde{A}}{\rho(\tilde{A})} = \tilde{u}.$$

Note that for any given vectors x and y, we have $xD_y = x \odot y = D_x y$. So

$$\tilde{u}D_w^{-1} = \tilde{u} \odot w^{-1} = \tilde{u} \odot \tilde{u}^{-1} \odot u = u,$$

From (20), we can conclude that

$$\tilde{u}\frac{\tilde{A}}{\rho(\tilde{A})} = \tilde{u}D_w^{-1}\frac{A}{\rho(A)}D_w = u\frac{A}{\rho(A)}D_w = uD_w = \tilde{u}.$$

This proves that $\tilde{u}$ is the maximal left eigenvector of $\tilde{A}$.
(b) Now we prove that $\tilde{v}$ is the maximal right eigenvector of $\tilde{A}$. From the definition of w and $\tilde{v}$, we know that

$$D_w\tilde{v} = w \odot \tilde{v} = \tilde{u} \odot u^{-1} \odot v \odot u \odot \tilde{u}^{-1} = v, \tag{23}$$

Hence

$$\frac{\tilde{A}}{\rho(\tilde{A})}\tilde{v} = D_w^{-1}\frac{A}{\rho(A)}D_w\tilde{v} = D_w^{-1}\frac{A}{\rho(A)}v = D_w^{-1}v = \tilde{v}.$$

□

We have proved that the optimal structure matrix defined by (22) has the predetermined target d-vector as the equilibrium. With $\tilde{A}$ and $\tilde{u}$, we get a new economic system. To understand the new system, classify products, and test system stability, it is natural to use the equilibrium state $u \odot v$ of the dual chain $Q_{\tilde{u}}$ (which was taken to be Q_u). $u \odot v$ is the maximal right eigenvector of the dual chain, and it is also the equilibrium solution of the invariant P derived from the original structure matrix. Thus, $\tilde{A}$ retained the invariant P from A. In fact, to classify products and test system stability for the optimal structure matrix $\tilde{A}$, we can simply return to P derived from the original structure matrix A; there is no need to use the dual chain $Q_{\tilde{u}}$. The proof is as follows. From the known $\tilde{v}$ and $w = \tilde{u} \odot u^{-1}$, we have proved in (23) that $w \odot \tilde{v} = v$, thus

$$\tilde{P}_\alpha = D_{\tilde{v}}^{-1}\frac{\tilde{A}_\alpha}{\rho(\tilde{A}_\alpha)}D_{\tilde{v}} = D_{\tilde{v}}^{-1}D_w^{-1}\frac{A_\alpha}{\rho(A_\alpha)}D_wD_{\tilde{v}} = D_v^{-1}\frac{A_\alpha}{\rho(A_\alpha)}D_v = P_\alpha.$$

In other words, we prove the following main property (i) of the optimal matrix.

Theorem 11.2. *Given a target d-vector $\tilde{u}$ and $\alpha \in [0,1)$, let $\tilde{A}$ be the optimal matrix determined by the above theorem. Then*

(i) *the invariant $\tilde{P}_\alpha$ of the optimal matrix $\tilde{A}_\alpha/\rho(\tilde{A}_\alpha)$ coincides with the invariant P_α of the original matrix $A_\alpha/\rho(A_\alpha)$.*

(ii) *the stability index (collapse time, collapse location) are the shared invariants of $A_\alpha, \tilde{A}_\alpha$, and P_α.*

Proof. We only need to prove the assertion (ii). The stability of $A_\alpha, \tilde{A}_\alpha$, and P_α are all realized through P_α. It follows from (i) that $A_\alpha, \tilde{A}_\alpha$ have the same invariant P_α. Thus, to prove (ii), we only need to show that A_α and P_α have the common stability index. This can be derived from the result below. More precisely, since we are concerned with when (i.e., n) and where (i.e., which product) a negative sign first appears in μ_n and x_n, we can omit the positive factors $\rho(A_\alpha)^n$ and v on both sides of the identity (24) below, and then we get the desired assertion. □

The following result is a natural extension of Theorem 5.6.

Theorem 11.3. (*Conversion Theorem/Equivalence Principle*) ([7]: *Theorem* 7.3) *The iterative sequence* $\{\mu_n\}_{n\geqslant 0}$ *of* P_α*, the iterative sequence* $\{x_n\}_{n\geqslant 0}$ *of* A_α*, and* v *satisfy the identity:*

$$\mu_n = \rho(A_\alpha)^n x_n \odot v, \qquad n \geqslant 0, \tag{24}$$

$$x_n = \rho(A_\alpha)^{-n} \mu_n \odot v^{-1}, \qquad n \geqslant 0. \tag{25}$$

Therefore, the two algorithms are equivalent.

If we take $x_0 = u$ in (24), then $\mu_0 = u \odot v$. Multiplying both sides of (24) by the vector $\mathbb{1}$ on the right side yields $\rho(A)^n x_n \odot v\mathbb{1} = \mu_n \mathbb{1}$. In particular, when $n = 0$, we get the normalization condition $uv = \mu_0 \mathbb{1} = 1 = \pi\mathbb{1}$. Note that μ_n and x_n differ by an exponential maximal order $\rho(A)^n$ and a constant vector factor v.

The above theorem also applies to the general transform $A \to \tilde{A}$ defined by (20).

12. **Programmable and intelligential efficient algorithms.** The main difference between Hua's economic optimization theory and other existing economic theories is that Hua's theory is computable and programmable. One of the important cornerstones of Chen's new economic optimization theory is the computation of the maximal eigenvalue of a non negative matrix and the corresponding maximal left and right eigenvectors. For system stability analysis, we not only need these three quantities but also need to know how to compute them accurately and efficiently.

Efficient algorithms play a significant role in Chen's new theory. In recent years, by using stochastics, Chen made many important progresses in efficient algorithms for matrix eigenpairs. It is generally believed that the power method converges too slowly and is of little use; also, the inverse power method, due to its use of the Rayleigh entropy estimate, may not be reliable. Chen introduced a new security estimate and completely avoided using the Rayleigh entropy estimate. By combining ideas from machine learning with these two methods and their variants, he achieved an universal high-speed algorithm.

Using our present computer and existing algorithms, it is not difficult to compute the maximal eigenvalues and the corresponding maximal left and right eigenvectors of low dimensional or symmetric matrices. However, accomplishing the same task for high-dimensional non-symmetric matrices is still very daunting. To address this problem, Chen proposed effective algorithms: **Matrix quasi-symmetrization technique** and **Eigenvector smoothing technique**. The former is used to reduce the amplitude of the matrix ($\max A - \min A = \max_{ij} a_{ij} - \min_{ij} a_{ij}$), and the latter is to make the right eigenvector v as flat as possible, i.e., $\max v / \min v = \max_i v^{(i)} / \min_i v^{(i)}$ is nearly constant.

Matrix quasi-symmetrization technique. Given a non negative irreducible matrix $A = (a_{ij})$, we define the following Q matrix (a matrix with non negative off diagonal elements and having zero-sum of each row):

$$Q = A - D_{A\mathbb{1}}.$$

Using irreducibility, we know that the equation

$$\mu Q = 0$$

has a unique positive solution $\mu = (\mu^{(1)}, \mu^{(2)}, \cdots, \mu^{(d)})$ satisfying the initial condition $\mu^{(1)} = 1$. With this positive vector μ, we can define the **quasi-symmetrized** matrix $\hat{A}$ of A:

$$\hat{A} = D_{\mu^{1/2}} A D_{\mu^{-1/2}}.$$

The important reason for introducing this concept is that A is symmetrizable with respect to μ, that is,

$$D_\mu A = A^* D_\mu \quad [= (D_\mu A)^*]$$

if and only if $\hat{A}$ is c. This is a very important generalization of symmetry. [3] illustrated the irreplaceable power of the quasi-symmetrization technique for computing the eigenpairs of non-symmetric but symmetrizable matrices through a simple tridiagonal matrix. We can not expect a general A to be symmetrizable, but $\hat{A}$ often reduces the amplitude of A.

Eigenvector Smoothing Technique. Both the power method and the inverse power method are mainly for computing eigenvectors of matrices. If the amplitude of the eigenvector is too large, it is simply impossible to achieve this using these two methods. Therefore, further transforming $\hat{A}$ into a matrix with an eigenvector as flat as possible (i.e., $\max v / \min v$ is nearly constant) is naturally beneficial for computation. At first glance, this problem seems difficult to approach, but with Chen's Theorem 5.4, the desired conclusion comes naturally.

Theorem 12.1. *Let w be any positive vector and define*

$$\bar{A} := A_w = D_w^{-1} A\, D_w = (w^{-1} \otimes w) \odot A,$$

where $w^{-1} \otimes w = \left(w^{(j)}/w^{(i)}\right)_{d\times d}$ is the tensor product of vector w^{-1} and vector w. Then $A_{\mathbb{1}} = A$. Denote the maximal right-eigenpair of A_w as $(\rho(A_w), g_w)$. Then we have

$$\rho(A_w) = \rho(A), \qquad g = g_{\mathbb{1}} = D_w g_w.$$

In particular, if $\max_i |w^{(i)} - v^{(i)}|$ is sufficiently small, then g_w is a vector close to a constant.

Proof. Since A_w is similar to A, it shares the same eigenvalues with A, and the eigenvector g_w satisfies

$$\rho(A) g_w = D_w^{-1} A D_w g_w.$$

Thus, we have $\rho(A) D_w g_w = A(D_w g_w)$, and hence the first assertion holds. Now, we prove the last assertion. Since

$$D_w g_w = g = D_v g_v = D_v \mathbb{1},$$

we have

$$g_w = D_w^{-1} D_v \mathbb{1} = D_{w^{-1}\odot v}\mathbb{1} = w^{-1} \odot v.$$

This leads to the desired assertion, as $w^{-1} \odot v$ is a vector close to a constant. □

Now we can state our eigenvector smoothing technique. Suppose w is an approximate solution of v, then the maximal right eigenvector of $\bar{A}$ is nearly a constant vector.

Chen's transform (the key transform $A \to P$) plays a central role in the theory. Altogether, 7 times are used in Chen's theory:

(i) The Proof of Hua's Fundamental Theorem in sections 3 and 5;
(ii) In stability testing, he used P instead of A to study the stability of the system;
(iii) In product grading and classification, he used the equilibrium solution of P, i.e., the component product of the maximal left and right eigenvectors of A, which has a clear economic significance;
(iv) He uses the dual chain twice in economic structure optimization, i.e., he uses the key transform twice;

(v) The stability test of the new optimal economic structure $\tilde{A}$ returns to P ($\tilde{P} = P$);

(vi) The stability test of the new economic structure $\kappa\tilde{A}$ after the secondary optimization still uses P, A, $\tilde{A}$, and $\kappa\tilde{A}$ with exactly the same stability as P.

From above, it can be undoubtedly seen that the transition probability matrix P is an invariant.

For popular videos about this article, please refer to the following website. http://rims.jsnu.edu.cn/f7/48/c16902a390984/page.htm.

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REFERENCES

[1] M.-F. Chen, *From Markov Chains to Non-Equilibrium Particle Systems*, 2^{nd} edition, World Scientific, Singapore, 2004.

[2] M.-F. Chen, *Eigenvalues, Inequalities, and Ergodic Theory*, Springer, London, 2005.

[3] M.-F. Chen, Hermitizable, isospectral complex matrices or differential operators, *Front. Math. China*, **13** (2018), 1267-1311.

[4] M.-F. Chen, New progress on L. K. Hua's optimization theory of economics (in Chinese), *Chin. J. Appl. Prob. Stat.*, **38** (2022), 159-178.

[5] M.-F. Chen, Economic ProductRank and quantum wave probability, *CSIAM Trans. Appl. Math.*, **6** (2025), 96-105.

[6] M.-F. Chen and Y.-H. Mao, *English Translation: Introduction to Stochastic Processes*, Higher Education Press and World Scientific, Singapore, 2021.

[7] M.-F. Chen, Y. C. Xie, B. Chen, Q. Zhou and T. Yang, *New Theory of Loo-Keng Hua's Economic Optimization and Demonstrational Examples (in Chinese)*, Beijing Normal University Publishing Group, Beijing, 2025.

[8] L. K. Hua, On the mathematical theory of globally optimal planned economic systems, *Proc. Nati. Acad. Sci. USA*, **81** (1984), 6549-6553.

[9] L. G. Hua, Mathematical theory of large scale optimization in planned economy, (IV) Mathematical model (utilizing the theory of contradictions), (V) Discussion of adjustments, (VI) Upper bounds on productive capacity, tables (in Chinese), *Kexue Tongbao (Chinese)*, **29** (1984), 961-965.

[10] L. G. Hua, Mathematical theory of large scale optimization in planned economy, (X) Crises in production systems (in Chinese), *Kexue Tongbao (Chinese)*, **30** (1985), 641-645.

[11] L. K. Hua, *Mathematical Theory of Large Scale Optimization in Planned Economy (in Chinese)*, China Finance and Economics Press, Beijing, 1987, Edited by M. F. Chen and H. K. Shi (new version), Beijing Normal University Publishing Group, Beijing, 2025.

[12] C. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, Philadephia, 2000.

[13] T. Yang, B. Chen and Q. Zhou, Demonstrational examples based on the new theory of L. K. Hua's economic optimization (in Chinese), *Chin. J. Appl. Prob. Stat.*, **40** (2024), 663-683.

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